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Optimal investment models with vintage capital: Dynamic programming approach

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ABSTRACT

The dynamic programming approach for a family of optimal investment models with vintage capital is here developed. The problem falls into the class of infinite horizon optimal control problems of PDE's with age structure that have been studied in various papers (Barucci and Gozzi, 1998, 2001; Feichtinger et al., 2003, 2006) either in cases when explicit solutions can be found or using Maximum Principle techniques.

The problem is rephrased into an infinite dimensional setting, it is proven that the value function is the unique regular solution of the associated stationary Hamilton–Jacobi–Bellman equation, and existence and uniqueness of optimal feedback controls is derived. It is then shown that the optimal path is the solution to the closed loop equation. Similar results were proven in the case of finite horizon by Faggian (2005b, 2008a). The case of infinite horizon is more challenging as a mathematical problem, and indeed more interesting from the point of view of optimal investment models with vintage capital, where what mainly matters is the behavior of optimal trajectories and controls in the long run.

Finally it is explained how the results can be applied to improve the analysis of the optimal paths previously performed by Barucci and Gozzi and by Feichtinger et al.

1. Introduction

The aim of the paper is to develop the dynamic programming (DP) approach for a family of optimal investment models with vintage capital, having infinite time horizon, with particular attention to the behavior of optimal paths in the long run.

We briefly recall that optimal investment models with vintage capital[1] have been studied in various papers in recent years, and modeled in various ways. That of optimal control of linear age structured equations is one of the possible approaches undertaken in literature. Such framework has been introduced in Barucci and Gozzi (1998, 2001) and then studied in various

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[1] For the study of vintage capital problems we recall also the papers (Barucci and Gozzi, 2001, 1998; Benhabib and Rustichini, 1991; Boucekkine et al., 2005; Chari and Hopenhayn, 1991; Fabbri and Gozzi, 2008).

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works, among which we highlight (Faggian, 2005b, 2008a; Faggian and Gozzi, 2004; Feichtinger et al., 2003, 2004, 2006). The optimal investment problem with vintage capital is there treated in two main cases:

- In Barucci and Gozzi (2001, 1998), Faggian and Gozzi (2004), Feichtinger et al. (2004) the production function is linear and the representative investor is price taker (corresponding to an objective function which is linear in the capital stock). The value function is then linear and the optimal investment strategies, together with the corresponding capital stock trajectories, can be explicitly calculated. Consequently, a deep qualitative analysis of the problem can be performed, including that of the long run behavior of the capital stock.
- In Feichtinger et al. (2006) the production function is linear and a large representative investor is considered (which leads to an objective function which is nonlinear in the capital stock). The value function is then nonlinear and the optimal investment strategies cannot be explicitly calculated. In Feichtinger et al. (2006) Feichtinger et al. make use of a particular version of Maximum Principle (first introduced in Feichtinger et al. (2003)) to analyze the optimal investment strategies, highlighting an anticipation effect. The paper does not analyze the long run behavior of the capital stock.

This second case is undertaken here too, as it is more interesting from the economic point of view, and yields challenging mathematical issues due to the lack of explicit solutions. Since the only paper with similar assumptions is Feichtinger et al. (2006), we clarify the contribution of the present work with respect to Feichtinger et al. (2006):

- We aim to study the *long run behavior of the optimal capital stock*, while in Feichtinger et al. (2006) the authors analyze the optimal investment dynamics (with no particular attention to properties in the long run);
- we use a DP approach, as the Maximum Principle appears less efficient in the study of optimal trajectories; this drives to the study of a family of Hamilton–Jacobi–Bellman (HJB from now on) equations that do not fit into the existing theory.

More precisely we develop the theory of a general mathematical model that may applies to a variety of examples and that encloses investment with vintage capital as a subcase. Such general mathematical problem is stated in Section 3.1. The main results are contained in Theorems 5.2 and 5.3 where it is shown that the Value Function is the unique solution of a suitable HJB equation, and in Theorem 5.4 where existence and uniqueness of optimal paths in closed loop form is proven.

Although the present work concerns mainly the theoretical matters, we would like to make clear that it adds both to mathematics and economics: on the one hand, Theorems 5.2 and 5.3 extend the existing theory of regular solutions of HJB equations in Hilbert spaces to a new set of problems; on the other hand, Theorem 5.4 allows to investigate the properties of the optimal state-control pairs (especially the long run behavior) in the vintage capital problem and in those other applications that can be framed into the same setting.

The paper is organized as follows. In Section 2 we introduce the model of investment with vintage capital, in Section 3 we rewrite the model as an abstract mathematical problem summarizing the main mathematical difficulties. We also review the existing literature on HJB equations in Hilbert spaces.

Then we come to the technical part. In Section 4 we introduce the notation, recall the definition of strong solution and the results on existence and uniqueness of strong solutions in the finite horizon case, as they appear in Faggian (2005b). In Section 5 we study the general mathematical problem and we state the main results. Proofs are postponed in Appendix A. In Section 6 we apply the results to optimal investment with vintage capital, and in Section 7 we drive the conclusions.

2. Optimal investment with vintage capital

We now describe the model of optimal investment with vintage capital, in the setting introduced by Barucci and Gozzi (2001, 1998), and later reprised and generalized by Feichtinger et al. (2003, 2004, 2006), and by Faggian (2005b, 2008a) and Faggian and Gozzi (2004).

The capital accumulation process is given by the following system

$$\begin{cases} \dfrac{\partial y(\tau,s)}{\partial \tau}+\dfrac{\partial y(\tau,s)}{\partial s}+\mu y(\tau,s)=u_1(\tau,s), \quad \tau>t, \ \ 0<s<\bar{s} \\ y(\tau,0)=u_0(\tau), \quad \tau>t \\ y(t,s)=x(s), \quad 0\le s\le \bar{s} \end{cases} \tag{2.1}$$

with $t \ge 0$ the initial time, $\bar{s} \in [0,+\infty]$ the maximal allowed age, and $\tau \in [0,T)$ with horizon $T=+\infty$. The unknown $y(\tau,s)$ represents the amount of capital goods of age s accumulated at time τ, the initial datum is a function $x(s)$ in the space[2] $L^2(0,\bar{s})$, $\mu>0$ is a depreciation factor. Moreover, $u_0:[t,+\infty)\to\mathbb{R}$ is the investment in new capital goods (u_0 is the boundary control) while $u_1:[t,+\infty)\times[0,\bar{s}]\to\mathbb{R}$ is the investment at time τ in capital goods of age s (hence, the distributed control). Investments are jointly referred to as the control $u=(u_0;u_1)$.

[2] $L^2(0,\bar{s})$ denotes the space of measurable and square integrable functions on $(0,\bar{s})$, and $\langle\cdot,\cdot\rangle_{L^2(0,\bar{s})}$ its scalar product.