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Stochastic viability and dynamic programming

Luc Doyen [a,*], Michel De Lara [b]

[a] CNRS, CERSP (UMR 7204 CNRS-MNHN-P6), Muséum National d'Histoire Naturelle, 55 rue Buffon, 75005 Paris, France

[b] Université Paris–Est, CERMICS, 6–8 avenue Blaise Pascal, 77455 Marne la Vallée Cedex 2, France

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ABSTRACT

This paper deals with the stochastic control of nonlinear systems in the presence of state and control constraints, for uncertain discrete-time dynamics in finite dimensional spaces. In the deterministic case, the viability kernel is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. In the present paper, we show how a stochastic viability kernel and viable feedbacks relying on probability (or chance) constraints can be defined and computed by a dynamic programming equation. An example illustrates most of the assertions.

1. Introduction

Risk, vulnerability, safety or precaution constitute major issues in the management and control of dynamical systems. Regarding these motivations, the role played by the acceptability constraints or targets is central, and it has to be articulated with uncertainty and, in particular, with stochasticity when a probability distribution is given. The present paper addresses the issue of state and control constraints in the stochastic context. For the sake of simplicity, we consider noisy control dynamic systems. This is a natural extension of deterministic control systems, which covers a large class of situations. Thus we consider the following state equation as the uncertain dynamic model

$$x(t+1) = f(t, x(t), u(t), w(t)),$$
$$t = t_0, \ldots, T-1, \quad \text{with } x(t_0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{X} = \mathbb{R}^n$ represents the system *state* vector at time t, $x_0 \in \mathbb{X}$ is the *initial condition* at *initial time* t_0, $u(t) \in \mathbb{U} = \mathbb{R}^p$ represents *decision* or *control* vector while $w(t) \in \mathbb{W} = \mathbb{R}^q$ stands for the *uncertain variable*, or *disturbance*, or *noise*.

The admissibility of decisions and states is first restricted by a non empty subset $\mathbb{B}(t, x)$ of admissible controls in $\mathbb{U}$ for all (t, x):

$$u(t) \in \mathbb{B}(t, x(t)) \subset \mathbb{U}. \tag{2}$$

Similarly, the relevant states of the system are limited by a non empty subset $\mathbb{A}(t, w(t))$ of the state space $\mathbb{X}$ possibly uncertain for all t,

$$x(t) \in \mathbb{A}(t, w(t)) \subset \mathbb{X}, \tag{3}$$

and a target

$$x(T) \in \mathbb{A}(T, w(T)) \subset \mathbb{X}. \tag{4}$$

We assume that

$$w(t) \in \mathbb{S}(t) \subset \mathbb{W}, \tag{5}$$

so that the sequences

$$w(\cdot) := (w(t_0), w(t_0+1), \ldots, w(T-1), w(T)) \tag{6}$$

belonging to

$$\Omega := \mathbb{S}(t_0) \times \cdots \times \mathbb{S}(T) \subset \mathbb{W}^{T+1-t_0} \tag{7}$$

capture the idea of possible *scenarios* for the problem. A scenario is an *uncertainty trajectory*.

These control, state or target constraints may reduce the relevant paths of the system (1). Such a feasibility issue can be addressed in a robust or stochastic framework. Here we focus on the stochastic case assuming that the domain of scenarios Ω is equipped with some probability $\mathbb{P}$. In this probabilistic setting, one can relax the constraint requirements (2)–(4) by satisfying the state constraints along time with a given confidence level β

$$\mathbb{P}(w(\cdot) \in \Omega \mid x(t) \in \mathbb{A}(t, w(t)) \text{ for } t = t_0, \ldots, T) \geq \beta \tag{8}$$

* Corresponding author.
E-mail address: luc.doyen@orange.fr (L. Doyen).

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by appropriate controls satisfying (2). Such probabilistic constraints are often called chance constraints in the stochastic literature as in [1,2]. We shall give proper mathematical content to the above formula in the following section. Concentrating now on motivation, the idea of stochastic viability is basically to require the respect of the constraints at a given confidence level β (say 90%, 99%). It implicitly assumes that some extreme events make irrelevant the robust approach [3] that is closely related to stochasticity with a confidence level 100%.

The problems of dynamic control under constraints usually refer to viability [4] or invariance [5,6] framework. Basically, such an approach focuses on inter-temporal feasible paths. From the mathematical viewpoint, most of viability and weak invariance results are addressed in the continuous time case. However, some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of the continuous dynamics as in [4,7]. Important contributions for discrete-time case are also captured by the study of the positivity for linear systems as in [8], or by the hybrid control as in [9,6] or [10]. Other references may be found in the control theory literature, such as [11,12] and the survey paper [13]. A large study focusing on the discrete-time case is also provided in [14].

Viability is defined as the ability to choose, at each time step, a control such that the system configuration remains admissible. The *viability kernel* associated with the dynamics and the constraints plays a major role regarding such issues. It is the set of initial states x_0 from which starts an acceptable solution. For a decision maker or control designer, knowing the viability kernel has practical interest since it describes the states from which controls can be found that maintain the system in a desirable configuration forever. However, computing this kernel is not an easy task in general. Of major interest is the fact that a dynamic programming equation underlies the computation or approximation of viability kernels as pointed out in [4,14].

The present paper aims at expanding viability concepts and results in the stochastic case for discrete-time systems. In particular, we adapt the notions of viability kernel and viable controls in the probabilistic or chance constraint framework. Mathematical materials of stochastic viability can be found in [15–17] but they rather focus on the continuous time case and cope with constraints satisfied almost surely. We here provide a dynamic programming and Bellman perspective for the probabilistic framework.

The paper is organized as follows. Section 2 is devoted to the statement of the probabilistic viability problem. Then, Section 3 exhibits the dynamic programming structure underlying such stochastic viability. An example is presented in Section 4 to illustrate some of the main findings.

2. The stochastic viability problem

Here we address the issue of state constraints in the probabilistic sense. This is basically related to risk assessment and management. This requires some specific tools inspired from the viability and invariance approach known for the certain case. In particular, within the probabilistic framework, we adapt the notions of viability kernel and viable controls.

2.1. Probabilistic assumptions and expected value

Probabilistic assumptions on the uncertainty $w(\cdot) \in \Omega$ are now added, providing a stochastic nature to the problem. Mathematically speaking, we suppose that the domain of scenarios $\Omega \subset \mathbb{W}^{T+1} = \mathbb{R}^q \times \cdots \times \mathbb{R}^q$ is equipped with a σ-field[1] $\mathcal{F}$ and a *probability* $\mathbb{P}$: thus, $(\Omega, \mathcal{F}, \mathbb{P})$ constitutes a *probability space*. The sequences

$$w(\cdot) = (w(0), w(1), \ldots, w(T-1), w(T)) \in \Omega$$

now become the *primitive random variables*.

Hereafter, we shall assume that the random process $w(\cdot)$ is independent and identically distributed (i.i.d.) under probability $\mathbb{P}$. In other words, we suppose that the probability is the product $\mathbb{P} = \bigotimes_{t=t_0}^{T} \mu$ of a common marginal distribution μ. The *expectation operator* $\mathbb{E}$ is defined on the set of measurable and integrable functions by

$$\begin{aligned}\mathbb{E}[g] &= \mathbb{E}_{\mathbb{P}}\left[g\left(w(\cdot)\right)\right] \\ &= \int_{\Omega} g(w(t_0), \ldots, w(T)) \mathrm{d}\mu(w(t_0)) \cdots \mathrm{d}\mu(w(T)),\end{aligned}$$

and we have that

$$\mathbb{E}_{\mathbb{P}}\left[g\left(w(t)\right)\right] = \mathbb{E}_{\mu}\left[g\left(w(t)\right)\right].$$

2.2. Controls and feedback strategies

It is well known that control issues in the uncertain case are much more complicated than in the deterministic case. In the uncertain context, we must drop the idea that the knowledge of open-loop decisions $u(\cdot) = (u(t_0), \ldots, u(T-1))$ induces one single path of sequential states $x(\cdot) = (x(t_0), \ldots, x(T))$. Open-loop controls $u(t)$ depending only upon time t are no longer relevant, contrarily to closed loop or feedback controls $\mathfrak{u}(t, x(t))$ which display more adaptive properties by taking into account the uncertain state evolution $x(t)$. In the stochastic setting, all the objects considered will be implicitly equipped with appropriate measurability properties. Thus we define a *feedback* as an element of the set of all measurable functions from the time-state pairs towards the controls:

$$\begin{aligned}\mathfrak{U} := \{\mathfrak{u} : (t, x) \in \{t_0, \ldots, T-1\} \\ \times \mathbb{X} \mapsto \mathfrak{u}(t, x) \in \mathbb{U}, \mathfrak{u} \text{ measurable}\}.\end{aligned} \tag{9}$$

The control constraint case restricts feedbacks to admissible feedbacks accounting for control constraints (2) as follows

$$\begin{aligned}\mathfrak{U}^{\mathrm{ad}} = \{\mathfrak{u} \in \mathfrak{U} \mid \mathfrak{u}(t, x) \in \mathbb{B}(t, x), \\ \forall (t, x) \in \{t_0, \ldots, T-1\} \times \mathbb{X}\}.\end{aligned} \tag{10}$$

Let us mention that, in the stochastic context, a feedback decision is also termed a *pure Markovian strategy*. Markovian means that the current state contains all the sufficient information of past system evolution to determine the statistical distribution of future states. Thus, only current state $x(t)$ is needed in the feedback loop among the whole sequence of past states $x(t_0), \ldots, x(t)$.

At this stage, we need to introduce some notations which will appear quite useful in what follows: the *state map* and the *control map*. Given a feedback $\mathfrak{u} \in \mathfrak{U}$, a scenario $w(\cdot) \in \Omega$ and an initial state x_0 at time $t_0 \in \{t_0, \ldots, T-1\}$, the solution state $x_f[t_0, x_0, \mathfrak{u}, w(\cdot)]$ is the state path $x(\cdot) = (x(t_0), x(t_0+1), \ldots, x(T))$ solution of the dynamics

$$x(t+1) = f(t, x(t), \mathfrak{u}(t, x(t)), w(t)), \quad t = t_0, \ldots, T-1$$

starting from the initial condition $x(t_0) = x_0$ at time t_0 and associated with feedback control $\mathfrak{u}$ and scenario $w(\cdot)$. The solution control $u_f[t_0, x_0, \mathfrak{u}, w(\cdot)]$ is the associated decision path $u(\cdot) = (u(t_0), u(t_0+1), \ldots, u(T-1))$ where $u(t) = \mathfrak{u}(t, x(t))$.

2.3. The stochastic viability kernel and viable feedbacks

The viability kernel plays a major role in the viability analysis. In the deterministic case, it is the set of initial states x_0 such that the

[1] For instance, $\mathcal{F}$ is the trace of Ω on the usual borelian σ-field $\mathcal{F} = \bigotimes_{t=t_0}^{T} \mathcal{B}(\mathbb{R}^q)$.