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Dynamic programming with ordered structures: Theory, examples and applications

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Abstract

The paper presents a dynamic, discrete optimization model with returns in ordered structures. It generalizes multiobjective methods used in vector optimization in two ways: from real vector spaces to ordered structures and from the static model to the dynamic model. The proposed methods are based on isotone homomorphisms. These methods can be applied in dynamic programming with returns in ordered structures. The provided numerical example shows an application of fuzzy numbers and random variables with stochastic dominance in dynamic programming. The paper also proposes applications in the following problems: a problem of allocations in the market model, a location problem, a railway routing problem, and a single-machine scheduling problem.

Keywords: Dynamic programming; Multiobjective programming; Ordered structures; Fuzzy numbers; Stochastic dominance

1. Introduction

This work considers the dynamic discrete decision making model with returns in a partially ordered set. To discuss this problem we first present Bellman's principle of optimality. Also, methods of narrowing the set of solutions will be described. These methods are analogical to multicriteria programming methods, but here—additionally—the dynamic aspect of the model must be taken into consideration. We show an approach based on the theory of isotone homomorphisms. Such a general approach, i.e. considering the structure of the partially ordered set, allows us to apply it to a number of mathematical structures which describe practical problems. Thus, as an interesting example we will present applications in some realistic problems: a problem of allocations in the market model, a location problem, a railway routing problem, and a single-machine scheduling problem.

Moreover, by using the presented model, we obtain the decision analysis tool in multicriteria problems. We can narrow a large set of efficient solutions to a small one or even to only one element. However, such a tool requires interactivity from the decision maker.

An optimal model involved in dynamic programming was introduced in 1957 by Bellman. Later, Bellman's principle of optimality was expanded to multicriteria dynamic programming with a lattical order [1] and to the preferences relation [2]. A similar approach to dynamic programming based on the relation (order) is used in this paper, as well.

The most recent contributions to dynamic programming in the view of the proposed approach are as follows. In the work of Bazgan et al. [3] we can find a quite similar approach to ordered structures in dynamic programming.

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The contribution of Bazgan is to use dominance relations in dynamic programming. The approach presented in this paper evaluates the mathematical techniques based on the so-called dominance relations (we call the objects featuring these relations ordered structures). The work of Mahdavi et al. [4] presents the problem of finding the shortest chain with the weights given by fuzzy numbers; the authors developed a dynamic programming approach for the fuzzy shortest chain problem using a fuzzy ranking method to avoid generating the set of nondominated paths, because the number of nondominated paths can be too large. The idea of Mahdavi et al. [4] is very similar to the idea proposed in our approach (narrowing methods), although we propose much more general methods which are applied not exclusively to fuzzy numbers. Linking different types of outcomes (as is done in the example provided in this paper) is the topic of the paper of Zaras [5], where three kinds of evaluations were considered: deterministic, stochastic, and fuzzy. Zaras [5] proposes the so-called mixed-data dominances to model the preferences with relation to each attribute, which is a very similar approach to the one proposed here. The paper of Toczylowski and Zoltowska [6] presents a pricing scheme for a multiperiod pool-based electricity action. This model is based on a mixed linear programming model that minimizes the sum of the compensation costs. The structure of this mathematical programming model (one criterion model) is a special case of the ordered structure presented in the approach discussed in this paper.

2. Notation and basic facts on dynamic programming

2.1. Dynamic system

Dynamics of the multiperiod system, consisting of T periods, is given by the state transformation

$$\Omega_t : D_t \to Y_{t+1} \quad \text{for } t = 1,2,\ldots,T,$$

where Y_t is a finite set of all feasible state variables at the beginning of the period $t \in \{1, 2, \ldots, T\}$; Y_{T+1} is a finite set of all states at the end of the process; $X_t(y_t)$ is a finite set of all feasible decision variables for the period t and the state $y_t \in Y_t$.

We will denote such process by P. On the basis of the above formulations we describe the following sets:

$D_t = \{d_t = (y_t, x_t) : y_t \in Y_t \wedge x_t \in X_t(y_t)\}$ is a set of all period realizations in the period $t \in \{1, 2, \ldots, T\}$.
$D = \{d = (d_1, \ldots, d_T) : \forall_{\{t\in\{1,\ldots,T\}} y_{t+1} = \Omega_t(y_t, x_t) \wedge x_T \in X_T(y_T)\}$ is a set of all process realizations d.
$D_t(y_t) = \{(y_t, x_t) : x_t \in X_t(y_t)\}$ is a set of all realizations in the period t which begin at the state $y_t \in Y_t$.
$d(y_t) = (y_t, x_t, \ldots, y_T, x_T)$ is a partial realization for given realization d which begins at the state $y_t \in Y_t$.
$D(y_t) = \{d(y_t) : d \in D\}$ is a set of all partial realizations which begin at the state $y_t \in Y_t$.
$D(Y_t) = \{D(y_t) : y_t \in Y_t\}$ is a set of all partial realizations which begin at the period t.

The presented deterministic and discrete dynamic process comes from Trzaskalik and Sitarz [7,8].

2.2. Ordered structures

We say that a triple $(W, \preccurlyeq, \circ)$ consisting of a set, a binary relation, and a binary operator, is an ordered structure, when

1. $(W,\preccurlyeq)$ is a partially ordered set (poset),
2. $\forall_{a,b,c\in W} \quad a\circ(b\circ c) = (a\circ b)\circ c$,
3. $\forall_{a,b,c\in W} \quad a\preccurlyeq b \Rightarrow a\circ c\preccurlyeq b\circ c \wedge c\circ a\preccurlyeq c\circ b$.

Relation $\preccurlyeq$ in the poset $(W, \preccurlyeq)$ generates relation $\prec$ in the following way:

$$a \prec b \quad \Leftrightarrow \quad a\preccurlyeq b \wedge a \neq b.$$

Additionally, if

3′. $\forall_{a,b,c\in W} \quad a \prec b \Rightarrow a\circ c \prec b\circ c \wedge c\circ a \prec c\circ b$,

then we call this ordered structure a strictly ordered structure.