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A dynamic programming algorithm for simulation of a multi-dimensional torus in a crossed cube ☆

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ABSTRACT

The torus is a popular interconnection topology and several commercial multicomputers use a torus as the basis of their communication network. Moreover, there are many parallel algorithms with torus-structured and mesh-structured task graphs have been developed. If one network can embed a mesh or torus network, the algorithms with mesh-structured or torus-structured can also be used in this network. Thus, the problem of embedding meshes or tori into networks is meaningful for parallel computing. In this paper, we prove that for $n \geqslant 6$ and $1 \leqslant m \leqslant \lceil n/2 \rceil - 1$, a family of 2^m disjoint k-dimensional tori of size $2^{s_1} \times 2^{s_2} \times \cdots \times 2^{s_k}$ each can be embedded in an n-dimensional crossed cube with unit dilation, where each $s_i \geqslant 2$, $\sum_{i=1}^{k} s_i = n - m$, and $\max_{1 \leqslant i \leqslant k}\{s_i\} \geqslant 3$ if n is odd and $m = \frac{n-3}{2}$; otherwise, $\max_{1 \leqslant i \leqslant k}\{s_i\} \geqslant n - 2m - 1$. A new concept, *cycle skeleton*, is proposed to construct a dynamic programming algorithm for embedding a desired torus into the crossed cube. Furthermore, the time complexity of the algorithm is linear with respect to the size of desired torus. As a consequence, a family of disjoint tori can be simulated on the same crossed cube efficiently and in parallel.

1. Introduction

The torus is a popular interconnection topology and several commercial multicomputers in use today use the torus as the basis of their communication network. It is also called a *wrap-around mesh* or a *toroidal mesh*. Many applications in scientific computing require the use of a multi-dimensional torus. For example, to study the theory of the strong nuclear force, known as quantum chromodynamics (QCD), a 5-dimensional torus connected multicomputer system was deployed at the Jefferson Lab in 2004, and a multicomputer system containing multiple 6-dimensional tori [2] is in construction at various sites such as the Brookhaven National Lab, Columbia University, and University of Edinburgh. Actually, torus-based interconnection networks are deployed in several high performance parallel computers. For instance, there are the iWrap [1] (torus), QCDOC (6D torus) [2], the MIT J-Machine [16](3D mesh), Cray T3D/T3E [17](3D torus), the Mosaic [18], Ametak 2010 [19], and the Tera Parallel Computer [20](torus). Moreover, there are many parallel algorithms with torus-structured and mesh-structured task graphs have been developed [14,15].

In addition to torus, several interconnection networks have been proposed for parallel computing and provide many good properties with compared to torus network. The k-ary n-cube is an n-dimensional torus with the same size, k, in each dimension, and the hypercube, which is a 2-ary n-cube; a mesh is a subgraph of a torus. The crossed cube proposed by Efe [5]

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is one of the most notable variations of hypercube, but some properties of the former are superior to those of the latter. For example, the diameter of the crossed cube is almost the half of that of the hypercube. With regard to embedding abilities of crossed cubes, many interesting results have received considerable attention [3,4,6–11,13,22–24]. In particular, Fan and Jia [6] found that a mesh of size $2 \times 2^{n-1}$ is a spanning subgraph of an n-dimensional crossed cube (CQ_n), and a family of two disjoint meshes of size $4 \times 2^{n-3}$ each can be embedded in an CQ_n with unit dilation. Recently, Lai and Tsai [12,21] proved that an n-dimensional Twisted cube and Möbius cube has the same result that of CQ_n, respectively. Dong et al. [3] proved that a family of two (four, respectively) disjoint meshes of size $2 \times 2 \times 2^{n-3}$ ($4 \times 2 \times 2^{n-5}$, respectively) each can be embedded in an CQ_n with unit dilation. In the same year, Dong et al. [4] also proved more general results that for $n \geqslant 4$ and $1 \leqslant m \leqslant \lfloor \frac{n}{2} \rfloor - 1$, a family of 2^m disjoint k-dimensional meshes of size $2^{s_1} \times 2^{s_2} \times \cdots \times 2^{s_k}$ each can be embedded in an CQ_n with unit dilation, where $\sum_{i=1}^{k} s_i = n - m$ and $\max_{1 \leqslant i \leqslant k}\{s_i\} \geqslant n - 2m - 1$. To our knowledge, the problem of how to embed k-dimensional tori into a crossed cube is open.

In this paper, we investigate the problem of embedding disjoint k-dimensional tori which the size of each dimension is power of 2. It is proved that for $n \geqslant 6$ and $1 \leqslant m \leqslant \lceil n/2 \rceil - 1$, a family of 2^m disjoint k-dimensional tori of size $2^{s_1} \times 2^{s_2} \times \cdots \times 2^{s_k}$ each can be embedded in an n-dimensional crossed cube with unit dilation, where each $s_i \geqslant 2$, $\sum_{i=1}^{k} s_i = n - m$ and $\max_{1 \leqslant i \leqslant k}\{s_i\} \geqslant 3$ if n is odd and $m = \frac{n-3}{2}$; otherwise, $\max_{1 \leqslant i \leqslant k}\{s_i\} \geqslant n - 2m - 1$. A new concept, *cycle skeleton*, is proposed to construct a dynamic programming algorithm for embedding a desired torus into the crossed cube. Furthermore, the time complexity of the algorithm is linear with respect to the size of desired torus. As a consequence, a family of disjoint tori can be simulated on the same crossed cube efficiently and in parallel.

The rest of this paper is organized as follows. Section 2 introduces notations, definitions and cycle skeletons in CQ_n that will be used in later. The k-dimensional mesh skeleton is described in first part of Section 3. Indeed, a linear algorithm is proposed to embed a mesh and torus into CQ_n. Conclusions are given in the final section.

2. Preliminaries

2.1. Notation and terminology

A topology of an interconnection network is conveniently represented with an undirected graph. The vertices of the graph represent the nodes of the network and each edge represents a communication link between two vertices. For definitions of relevant concepts from graph theory and interconnection networks we refer the reader to [15]. Two subgraphs of G are *vertex-disjoint* (or *disjoint* for short) if they are no vertex in common. A walk in a graph is a finite sequence $\omega = \mathbf{v}_0, e_1, \mathbf{v}_1, e_2, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, e_k, \mathbf{v}_k$ whose terms are alternately vertices and edges such that, for $1 \leqslant i \leqslant k$, the edge e_i has ends $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$, thus each edge e_i is immediately preceded and succeeded by the two vertices with which it is incident. In particular, a walk ω is called a path if all vertices, $\mathbf{v}_i$ for $0 \leqslant i \leqslant k$, of the walk ω are distinct where $\mathbf{v}_0$ and $\mathbf{v}_k$ are called end-vertices of the path ω. For simplicity, the path ω is also denoted by $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. If $\mathbf{v}_0 = \mathbf{v}_k$, then ω is called a cycle. A cycle of length l is called a l-cycle.

While the problem of embedding one interconnection network into another is considered, a one-to-one mapping ϕ from $V(G)$ to $V(H)$ denote an embedding of one guest graph, G, into another host graph, H. An edge $(\mathbf{u},\mathbf{v})$ of G corresponds to a path in H from node $\phi(\mathbf{u})$ to node $\phi(\mathbf{v})$. The dilation is one of most important parameters of an embedding and expressed by the maximum length of a path of the host graph that is mapped by an edge of the guest graph.

Throughout this paper, n always is a positive integer while Z_n denotes the set $\{0,1,\ldots,n-1\}$. To define an n-dimensional crossed cube, denoted as CQ_n, the relation so called "*pair related*" defined on 2-bit binary strings is introduced. Two binary strings $\mathbf{u} = u_1u_0$ and $\mathbf{v} = v_1v_0$, are pair related, denoted as $\mathbf{u} \sim \mathbf{v}$, if and only if $(\mathbf{u}, \mathbf{v}) \in R$ where $R = \{(00,00), (10,10), (01,11), (11,01)\}$.

Let $B_2 = \{u_1u_0 : u_i = 0 \text{ or } 1\}$ and $\gamma : B_2 \to B_2$ be a bijection mapping, which is defined as follows: $\gamma(u_1u_0) = v_1v_0$ if and only if $(u_1u_0, v_1v_0) \in R$. Let $\mathcal{C}_b : B_2 \to B_2$ and $\mathcal{C}_f : B_2 \to B_2$ be two bijection mappings, which are defined as follows $\mathcal{C}_b(u_1u_0) = u_1\overline{u_0}$ and $\mathcal{C}_f(u_1u_0) = \overline{u_1}u_0$, respectively. Thus, one obtains a function $g \circ f : B_2 \to B_2$ defined by $(g \circ f)(\mathbf{u}) = g(f(\mathbf{u}))$ for all $\mathbf{u} \in B_2$ where $g,f \in \{\gamma, \mathcal{C}_f, \mathcal{C}_b\}$. The following proposition is fundamental and useful in the proof of Lemma 5.

Proposition 1. *For $k \geqslant 1$, let $\mathbf{u}$ be any 2-bit binary string. Then,*

(1) $f^{2k}(\mathbf{u}) = \mathbf{u}$ *where* $f \in \{\gamma, \mathcal{C}_f, \mathcal{C}_b\}$,
(2) $\mathcal{C}_f \circ \gamma(\mathbf{u}) = \gamma \circ \mathcal{C}_f(\mathbf{u})$, *and*
(3) $\mathcal{C}_b \circ \gamma(\mathbf{u}) = \mathcal{C}_f \circ \gamma \circ \mathcal{C}_b(\mathbf{u})$.

Subsequently, a crossed cube of dimension n, denoted CQ_n, is defined recursively as follows.

Definition 1 [5]. The crossed cube CQ_1 is a complete graph with two vertices labeled by 0 and 1, respectively. For $n \geqslant 2$, an n-dimensional crossed cube CQ_n consists of two $(n-1)$-dimensional crossed cubes, CQ^0_{n-1} and CQ^1_{n-1}, and a perfect matching between the vertices of CQ^0_{n-1} and CQ^1_{n-1} according to the following rule: Let $V(CQ^0_{n-1}) = \{0u_{n-2}u_{n-3}\cdots u_0 : u_i = 0 \text{ or } 1\}$ and $V(CQ^1_{n-1}) = \{1v_{n-2}v_{n-3}\cdots v_0 : v_i = 0 \text{ or } 1\}$. The vertex $\mathbf{u} = 0u_{n-2}u_{n-3}\cdots u_0 \in V(CQ^0_{n-1})$ and the vertex $\mathbf{v} = 1v_{n-2}v_{n-3}\cdots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if