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A numerical method for hybrid optimal control based on dynamic programming

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ABSTRACT

This contribution extends a numerical method for solving optimal control problems by dynamic programming to a class of hybrid dynamic systems with autonomous as well as controlled switching. The value function of the hybrid control system is calculated based on a full discretization of the state and input spaces. A bound for the error due to discretization is obtained from modeling the error as perturbation of the continuous dynamics and the cost terms. It is shown that the bound approaches zero and that the value function of the discretized variant converges to the value function of the original problem if the discretization parameters go to zero. The performance of a numerical scheme exploiting the discretized system is illustrated for two different examples treated previously in literature.

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1. Introduction

The theory of optimal control of dynamic systems is mainly based on two principles, namely the maximum principle (MP) and dynamic programming (DP), see e.g. [1]. The MP provides necessary conditions for optimal trajectories and co-states while the DP theory derives the value function as the solution of a partial differential equation. The optimal input trajectory is calculated for both cases by solving a static optimization problem along the evolution of the system. The main elements of this optimization problem, the co-states (MP) or the value function (DP) respectively, are calculated off-line. The co-states are valid only for a particular initial state, while the value function covers the whole state space. Thus, the evaluation of the necessary conditions of the MP results in an optimal open-loop control and applying the DP principle leads to optimal closed-loop control.

Both theories have been extended to hybrid dynamical systems, which combine continuous dynamics with discrete events, as e.g. recently described in the survey paper [2]. With respect to extending the MP to hybrid systems, necessary optimality conditions for the co-states at the switching times are derived in [3–5]. The extension of DP to hybrid systems, as considered in [6,7], determines the value function as the solution of a set of partial differential equations.

Algorithms exploiting these principles are confronted with a number of problems. For relatively "simple" classes of systems, like linear continuous dynamics and quadratic costs, approximate closed-loop solutions exist, see e.g. [8] for autonomous switching, [9] for controlled switching, or [10] dealing with discrete time linear hybrid systems. Algorithms for applying the MP to general hybrid optimal control problems, solve the optimal control problem for a given sequence of discrete states and repeat this in a combinatorial sense over all possible discrete state sequences (See [11]). Such procedures typically incur exponential complexity with respect to the number of discrete states. For DP, the main contribution to computational complexity results from the discretization of the continuous part of the dynamics. For example, the prototype algorithm described in [12] uses a discretization of the hybrid state space and computes an approximation of the value

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function for the discretized system. In [13], the calculation of the value function for the discretized system is formulated as a linear optimization problem.

A number of approaches have been proposed to reduce the computational complexity of determining optimal controls for hybrid systems. For example in [14], optimality zones for particular dynamics are pre-computed by sampling and a brute force method, such that the remaining set of discrete sequences is smaller. In [15,16], branch and bound methods are used to prune the search tree of sequences. In [17], the discrete states are relaxed, leading to mixed continuous dynamics. Algorithms implementing an efficient calculation, like presented in [18,19], of the value function of hybrid systems are rare.

The work proposed here considers a class of hybrid models with autonomous (forced) as well as controlled (optional) switching with possible state resets. The algorithm presented for the solution is based on the DP principle and on a full discretization of the hybrid optimal control problem. The base of the algorithm is a Semi-Lagrangian discretization scheme in combination with piecewise affine functions to approximate the value function. The scheme, which is motivated by the one presented in [20,21] for purely continuous systems, essentially discretizes in space and time to obtain a finite representation of the hybrid dynamics. The value function for the discretized system, which is interpreted as a perturbed discrete time process, is linearly interpolated over the continuous state space to obtain an approximation of the value function for the original system. While the basic algorithm has been presented already in [22], the main contribution of this paper is the convergence analysis for the general numerical scheme. For the convergence analysis, the state space discretization is interpreted as a perturbation of the continuous dynamics and the costs terms. The approach is stimulated by the work in [23], which models the numerical error of computing the solution of an optimal stopping problem for continuous systems as perturbation.

The optimal control problem considered here is formulated as an exit time optimal control problem. As discussed later, this setting includes a large variety of different optimal control problems. Unlike an infinite time discounted optimal control problem (see [24–26]), the region of computation need not be invariant.

The paper is organized as follows. After some preliminaries, the hybrid system is defined in Section 3, and the optimal control problem is introduced in Section 4. In Section 5 the discrete time process under perturbation is defined and estimates for the difference between the original and the perturbed value function are provided. The state space discretization, its characterization as pertubation, and the algorithm for solving the fully discretized problem are described in Section 6. Section 7 defines the control law based on the calculated value function, and the approach is demonstrated for two examples in Section 8. The paper concludes with a discussion of the performance, the limits, and possible extensions in Section 9.

2. Preliminaries

The following notation is used throughout the paper:

- $|x| := (x^\mathsf{T} x)^{1/2}$ denotes the Euclidean norm of $x \in \mathbb{R}^n$.
- int S, ∂S, and $\bar{S}$ denote the interior, the boundary and closure of the set S. The complement of the set $S \subset \mathbb{R}^n$ relative to $\mathbb{R}^n$ is denoted by $S^c := \mathbb{R}^n \backslash S$.
- $B(\bar{x}, r) := \{x \in \mathbb{R}^n \mid |x - \bar{x}| < r\}$ denotes an open ball with the radius r and the center in $\bar{x}$. Let S be a set, then with slight abuse of notation we write $B(S, r) := \cup_{x \in S} B(x, r)$.
- $d(x, S) := \inf_{\bar{x} \in S} |x - \bar{x}|$ is the minimum distance from x to S, while $d(S^*, S) := \inf_{x \in S^*} d(x, S)$ denotes the minimum distance between two points in the sets S and S^*.
- The *index function* with respect to a set $S \subset \mathbb{R}^n$ is defined by:

$$\mathit{l}_S(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \in S^c. \end{cases}$$

3. Hybrid dynamical systems

The hybrid control system is defined similar to [6] but without time delays in the transitions and without inputs to control autonomous transitions.

3.1. Model definition

A *hybrid control system* $\mathcal{H}$ is defined as the tuple:

$$\mathcal{H} := (Z, X, U, f, E, r, G, q, W), \tag{1}$$

where the following applies for the components:

Assumption A1. Assumptions for the hybrid system $\mathcal{H}$:

- $Z = \{1, 2, \ldots, n_Z\}$ is the finite set of discrete states called *locations*.
- $X = \{X_z\}_{z \in Z}$ is the set of continuous state spaces, with X_z as the closure of a connected open subset of $\mathbb{R}^{n_x}$.