Reliability Engineering and System Safety 79 (2003) 139–148

RELIABILITY ENGINEERING & SYSTEM SAFETY

www.elsevier.com/locate/ress

Sensitivity analysis in Gaussian Bayesian networks using a symbolic-numerical technique

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Abstract

The paper discusses the problem of sensitivity analysis in Gaussian Bayesian networks. The algebraic structure of the conditional means and variances, as rational functions involving linear and quadratic functions of the parameters, are used to simplify the sensitivity analysis. In particular the probabilities of conditional variables exceeding given values and related probabilities are analyzed. Two examples of application are used to illustrate all the concepts and methods.

Keywords: Sensitivity; Gaussian models; Bayesian networks

1. Introduction

Sensitivity analysis is becoming an important and popular area of work. Plain outputs of mathematical models are often insufficient for practical decision-making; the outputs must be further evaluated before a solid ground for decision-making has been established. To evaluate a model output, the model can be exposed to sensitivity analysis, indicating how sensitive the output is to change in values of model parameters [4–8].

In some cases, the choice of parameter values has an extreme influence on the model outputs. For example, it is well known how sensitive the distributional assumptions and parameter values are to tail distributions (see Refs. [2,9,10,14]). If this influence is neglected, the consequences can be disastrous. Thus, sensitivity analysis plays a very important role in model validation and model output evaluation.

Laskey [17] seems to be the first to address the complexity of sensitivity analysis of Bayesian networks. She introduced a method for computing the partial derivative of a posterior marginal probability with respect to a given parameter. Castillo et al. [5,4] show that the function expressing the posterior probability is a quotient of linear functions in the parameters and the evidence values in the discrete case, and of the means, variances, and evidence values, but covariances can appear squared. This discovery allows simplification of sensitivity analysis and makes it computationally efficient (see, for example, Ref. [16] or Ref. [12]).

In this paper we address the problem of sensitivity analysis in Gaussian Bayesian networks and show how changes in the parameter and evidence values influence marginal and conditional probabilities given the evidence.

The paper is structured as follows. In Section 2 we briefly review Gaussian Bayesian networks and introduce our working example. In Section 3 we discuss how to perform exact propagation in Gaussian Bayesian networks. Section 4 is devoted to symbolic propagation. Section 5 analyses the sensitivity problem. Section 6 presents the damage of concrete structures example. Finally, in Section 7 we make some concluding remarks.

2. Gaussian Bayesian network models

In this section we briefly review Bayesian network models.

Definition 1 (Bayesian network). A Bayesian network is a pair $(\mathcal{G},\mathcal{P})$, where $\mathcal{G}$ is a directed acyclic graph (DAG), $\mathcal{P}=\{p(x_1|\pi_1),\dots,p(x_n|\pi_n)\}$ is a set of n conditional probability densities (CPD), one for each variable, and Π_i is the set of parents of node X_i in $\mathcal{G}$. The set P defines

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PII: S0951-8320(02)00225-9

the associated joint probability density as

$$p(\mathbf{x}) = \prod_{i=1}^{n} p(x_i|\pi_i). \quad (1)$$

The two main advantages of Bayesian networks are: (a) the factorization implied by Eq. (1), and (b) the fact that conditional independence relations can be inferred directly from the graph $\mathcal{G}$ [3,13,21].

Definition 2 (Gaussian Bayesian network). A Bayesian network is said to be a Gaussian Bayesian network if and only if the JPD associated with its variables X is a multivariate normal distribution, $N(\mu, \Sigma)$, i.e. with joint probability density function

$$f(x)=(2\pi)^{-n/2}|\Sigma|^{-1/2}\exp\{-1/2(x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu)\}, \quad (2)$$

where μ is the n-dimensional mean vector, Σ is the $n \times n$ covariance matrix, $|\Sigma|$ is the determinant of Σ, and μ^{T} denotes the transpose of μ.

Gaussian Bayesian networks have been treated, among others, by Kenley [15], Shachter and Kenley [22]), and Castillo et al. [3]. The JPD of the variables in a Gaussian Bayesian network can be specified as in Eq. (1) by the product of a set of CPDs whose joint probability density function is given by

$$f(x_i|\pi_i) \sim N\left(\mu_i + \sum_{j=1}^{i-1} \beta_{ij}(x_j - \mu_j), v_i\right), \quad (3)$$

where β_{ij} is the regression coefficient of X_j in the regression of X_i on the parents of X_i, Π_i, and

$$v_i = \Sigma_i - \Sigma_{i\Pi_i}\Sigma_{\Pi_i}^{-1}\Sigma_{i\Pi_i}^{\mathrm{T}}$$

is the conditional variance of X_i, given $\Pi_i = \pi_i$, where Σ_i is the unconditional variance of X_i, $\Sigma_{i\Pi_i}$ is the covariances between X_i and the variables in Π_i, and Σ_{Π_i} is the covariance matrix of Π_i. Note that β_{ij} measures the strength of the relationship between X_i and X_j. If $\beta_{ij} = 0$, then X_j is not a parent of X_i.

Note that while the conditional mean $\mu_{x_i}|\pi_i$ depends on the values of the parents π_i, the conditional variance does not depend on these values. Thus, the natural set of CPDs defining a Gaussian Bayesian network is given by a collection of parameters $\{\mu_1,\dots,\mu_n\}$, $\{v_1,\dots,v_n\}$, and $\{\beta_{ij}|j < i\}$, as shown in Eq. (3).

The following is an illustrative example of a Gaussian Bayesian network.

Example 1 (Gaussian Bayesian network). Assume that we are studying the river in Fig. 1(a), where we have indicated the four cross-sections A, B, C and D, where the water discharges are measured. The mean time of the water going from A to B and from B to D is one day, and the mean time from C to D is two days. Thus, we register the set (A, B, C, D) with the corresponding delays. Assume that the joint water discharges can be assumed to be normal distributions and that we are interested in predicting B and D, one and two days later, respectively, from the observations of A and C.

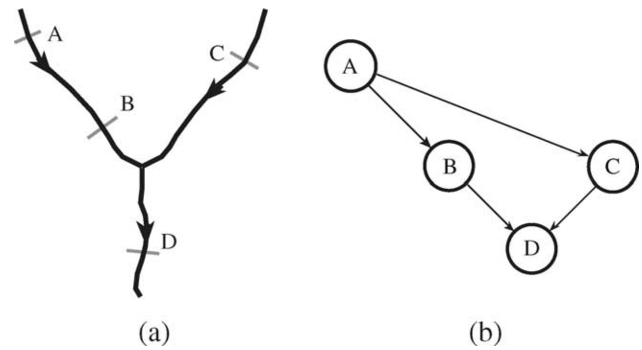

Fig. 1. (a) The river in Example 1 and the selected cross-sections, and (b) the graph of the Gaussian Bayesian network used to solve the problem.

In Fig. 1(b) we have shown the graph associated with a Gaussian Bayesian network that shows the dependence structure of the variables involved.

Suppose that the random variable (A, B, C, D) is normally distributed, i.e. $\{A,B,C,D\} \sim N(\mu, \Sigma)$. A Gaussian Bayesian network is defined by specifying the set of CPDs appearing in factorization (1), which gives

$$f(a,b,c,d) = f(a)f(b|a)f(c|a)f(d|b,c), \quad (4)$$

where

$$f(a) \sim N(\mu_A, v_A), \quad (5)$$

$$f(b|a) \sim N(\mu_B + \beta_{BA}(a - \mu_A), v_B),$$

$$f(c|a) \sim N(\mu_C + \beta_{CA}(a - \mu_A), v_C),$$

$$f(d|b,c) \sim N(\mu_D + \beta_{DB}(b - \mu B) + \beta_{DC}(c - \mu_C), v_D).$$

The parameters involved in this representation are $\{\mu_A, \mu_B, \mu_C, \mu_D\}$, $\{v_A, v_B, v_C, v_D\}$, and $\{\beta_{BA}, \beta_{CB}, \beta_{DB}, \beta_{DC}\}$.

Note that so far, all parameters have been considered in symbolic form. Thus, we can specify a Bayesian model by assigning numerical values to the parameters above. For example, for

$$\mu_A = 3, \quad \mu_B = 4, \quad \mu_C = 9, \quad \mu_D = 14,$$

$$v_A = 4, \; v_B = 1, \; v_C = 4, \; v_D = 1;$$

$$\beta_{BA} = 1, \; \beta_{CA} = 2, \; \beta_{DB} = 1, \; \beta_{DC} = 1,$$

we get

$$\mu = \begin{pmatrix} 3 \\ 4 \\ 9 \\ 14 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 4 & 8 & 12 \\ 4 & 5 & 8 & 13 \\ 8 & 8 & 20 & 28 \\ 12 & 13 & 28 & 42 \end{pmatrix}.$$