Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Materials and Design 25 (2004) 663–672
Materials & Design
www.elsevier.com/locate/matdes

Sensitivity analysis of heat conduction for functionally graded materials

Biaosong Chen, Liyong Tong *

School of Aerospace, Mechanical and Mechatronic Engineering, University of Sydney, NSW 2006, Australia

Received 3 September 2003; accepted 11 March 2004
Available online 22 April 2004

Abstract

A sensitivity analysis is presented for the steady-state and transient heat conduction of functionally graded materials (FGMs). Based on the finite element method, the sensitivity equations of heat conduction are presented by using the direct method and the adjoint method. In the solution of transient problem, the precise time integration (PTI) is employed. The spatial volume fractions of materials of FGM (size problem) and the shape design parameters are considered. Detailed formulations especial for the FGMs are provided. The numerical examples are presented to demonstrate the precision and applicability of the proposed method.

Keywords: (A) composite; (G) thermal analysis; Functionally graded materials

1. Introduction

Compared to the conventional materials, the functionally graded materials (FGMs) show their predominant advantages of thermal and structural performance in high-temperature environments. During the past two decades, much research of FGMs have been focused on the manufacture, material design and property estimation, thermal and structural analysis [1,2]. As the role of FGMs in industrial applications is further enhanced, attentions are drawn, recently, to design optimization of FGMs [3]. One of the basic techniques of design optimization is sensitivity analysis. Sensitivities are valuable information for design optimization, stochastic analysis and inverse problems. They reflect the change rates (derivatives) of system responses with respect to design variables. In this paper, the sensitivity analysis of heat conduction will be discussed, which will serve as the basis for the research of coupled thermo-structural responses (thermal stress, thermal deformation, thermal buckling and thermal vibration).

Numerous publications on sensitivity analysis of heat conduction for general problems have presented the development history of the relevant research. Haftka and Malku [4,5] presented the sensitivity analysis of heat conduction based on a discrete model, which considered both the linear and nonlinear cases. Dems [6,7] discussed the sensitivity analysis of size and shape design variable for linear steady/transient heat conduction. Meric [8] proposed a BEM method of shape sensitivity analysis for steady nonlinear heat conduction problem. Tortorelli et al. [9,10] studied the sensitivity analysis of linear/nonlinear heat conduction based on continuum model. They also presented the finite element formulations for solving the sensitivity equations. Kleiber and Sluzalec [11] deduced the sensitivity analysis equations of shape design variables by employing the material derivative and control volume methods. In conjunction with the semi-analytical method and finite element method, Gu and Grandhi [12] investigated the sensitivity analysis of transient heat conduction and discussed the adjoint method and direct method. Dems and Rousselet [13,14] studied the sensitivity analysis of shape design for transient thermal problem. Blackwell et al. [15] carried out the sensitivity analysis with respect to material properties and initial/boundary parameters based on the energy equations. Lee and Kwak [16] performed shape

* Corresponding author. Tel.: +61-2-9351-6949; fax: +61-2-9351-4841.
E-mail address: ltong@aeromech.usyd.edu.au (L. Tong).

0261-3069/$ - see front matter
doi:10.1016/j.matdes.2004.03.007

Nomenclature

$\boldsymbol{A}$	matrix exponential	t	time
$\boldsymbol{B}$	derivatives of element shape function	$\boldsymbol{T}$	nodal temperature
$\boldsymbol{c}$	material capacity	V_c	volume fraction of ceramic
$\boldsymbol{H}$	system coefficient matrix	V_m	volume fraction of metal
k	time step index	$\boldsymbol{v}$	system state vector
$\boldsymbol{k}$	thermal conductivity tensor	$\dot{\boldsymbol{v}}$	velocity of system vector
$\boldsymbol{K}$	heat conductance matrix	$\dot{\boldsymbol{T}}$	rate of nodal temperature
$\boldsymbol{M}$	heat capacity matrix	x	design variable
$\boldsymbol{N}$	element shape function	$\boldsymbol{\Lambda}$	adjont vector
n^e	number of element nodes	τ	time step length
$\boldsymbol{r}$	time-dependent exerted vector	ρ	material density
$\boldsymbol{R}$	heat load vector		

design sensitivity analysis using boundary integral equation formulation. The authors in [17–19] investigated the application of precise time integration (PTI) to the transient analysis and its sensitivity for heat conduction. The PTI method possesses high numerical precision and some unique numerical merits. Numerical results showed that the PTI method is superior to the conventional time difference method in respect of numerical precision. The work of topological sensitivity analysis on heat conduction is presented by Novotny et al. [20]. In their research, the topological sensitivity is defined in a specific mathematical framework so that the numerical techniques of shape sensitivity analysis can be applied.

In the field of sensitivity analysis for design optimization of FGMs, there exist a limited number of publications. Tanaka et al. [21,22] studied the design optimization of thermal stress of FGM, in which the direct method was employed to compute sensitivities based on the incremental form of governing equations. Cho and Ha [23] used global finite difference method to carry out the sensitivity analysis in the design optimization of two-dimensional FGM problems. Turteltaub [24,25] investigated the design optimization and optimal control of FGMs. The sensitivity analysis of his work was based on continuum model in conjunction with adjoint method.

In this paper, the sensitivity analysis is carried out based on the discrete form of finite element equations. The direct method and the adjoint method are employed to derive the sensitivity equations for both steady-state and transient heat conduction problem. In the solution of transient problem, the PTI is applied. The graded method [26], which computes the material properties of FGM by using the generalized isoparametric formulations, is employed in order to improve the modeling capability and numerical precision. Detailed formulations of sensitivity analysis are provided. Numerical examples are presented to validate the accuracy and the applicability of the method.

2. Sensitivity analysis of heat conduction

2.1. Steady-state problem

The finite element equations of steady-state heat conduction can be expressed as

$$\boldsymbol{K}\boldsymbol{T} = \boldsymbol{R}, \tag{1}$$

where $\boldsymbol{K}$, $\boldsymbol{T}$, $\boldsymbol{R}$ are conduction matrix, nodal temperature vector and thermal load vector, respectively. Suppose that in the optimal model, we have an index function that can be viewed as a constraint function or an objective function and has the expression

$$g = g(x, \boldsymbol{T}), \tag{2}$$

where x is the design variable (here only one design variable is considered, however, the following formulations are valid for the problem of multi-design variables), then its sensitivity with respect to the design variable x can be expressed as

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial \boldsymbol{T}}\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}x}. \tag{3}$$

There are two methods to compute the sensitivity $\mathrm{d}g/\mathrm{d}x$: the direct method and the adjoint method.

(1) Direct method: By differentiating Eq. (1), we have

$$\boldsymbol{K}\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}x} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}x} - \frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}x}\boldsymbol{T}. \tag{4}$$

After solving the above equations, we can substitute the results of $\mathrm{d}\boldsymbol{T}/\mathrm{d}x$ into Eq. (3) to determine the sensitivity $\mathrm{d}g/\mathrm{d}x$.

(2) Adjoint method: Firstly, we introduce the adjoint vector $\boldsymbol{\Lambda}$ which satisfies

$$\boldsymbol{K}\boldsymbol{\Lambda} = \left(\frac{\partial g}{\partial \boldsymbol{T}}\right)^{\mathrm{T}}. \tag{5}$$

Then the sensitivity of index function can be obtained by

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\partial g}{\partial x} + \boldsymbol{\Lambda}^{\mathrm{T}}\left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}x} - \frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}x}\boldsymbol{T}\right). \tag{6}$$