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A Sensitivity Analysis of Matching Coin Game Strategies

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Abstract—Consider a matching game where two players (labeled Player #1 and Player #2) each have a coin and for each trial, shows either a head or a tail. When either a head-tail or a tail-head is shown, Player #1 wins a specified amount; otherwise, Player #2 wins a specified amount. There exists a Nash equilibrium that is stable with respect to an individual player's deviation from it, hence, it is not advantageous for any of the players to use a strategy different from the optimal strategy, since the value of the player's expected winnings would not be increased. This note describes strategies for this game and shows the limitations of each player in securing guaranteed expected winnings. A generalization of this game to three people is discussed and analyzed. The primary contribution is a sensitivity analysis of the expected winnings with respect to small perturbations in the winnings parameters for this generalized game.

Keywords—Games, Strategy, Equilibrium, Sensitivity Analysis, Probability.

1. MATCHING COIN GAME DESCRIPTION AND BACKGROUND

Marilyn vos Savant writes a weekly column, *Ask Marilyn*, which appears in Parade Magazine, a supplement to several Sunday newspapers in the United States. In the March 31, 2002 column, a simple matching game was described [1]. The zero-sum game involves two players that are each given a fair coin labeled with a head and a tail. Each player can show either a head or a tail. If both players show a head, then the second player (referred to as Player #2) wins \$3. If both players show a tail, then Player #2 wins \$1. If one player shows a head and the other shows a tail, then Player #1 wins \$2. Clearly, if both players flip their coins, and the coin is fair (hence, randomize showing a head), then the game is fair (i.e., the expected winnings for both Players #1 and #2 is zero). In the April 7, 2002 *Ask Marilyn* column, vos Savant proposes that Player #1

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show tails twice as often as heads, resulting in Player #1 winning on average \$1 for every six plays of the game [2].

In the June 2002 issue of *OR/MS Today*, as well as the June 2003 issue of *SIAM News*, Vasko and Newhart identify the flaw in this response, and provide the optimal strategy for Player #2 of randomly showing a head with probability 3/8 based on elementary game theory [3,4]. To see this, define $W1$ = the winnings for Player #1, $x_1 = P$ {Player #1 shows a head}, and $x_2 = P$ {Player #2 shows a head}. Then, $E[W1]$, the expected winnings for Player #1, is

$$\begin{aligned} E[W1] &= -(\$3)\,x_1x_2 - (\$1)\,(1-x_1)\,(1-x_2) + (\$2)\,(x_1\,(1-x_2) + x_2\,(1-x_1)) \\ &= -(\$8)\,x_1x_2 + (\$3)\,(x_1) + (\$3)\,(x_2) - \$1 \\ &= (-(\$8)\,x_1 + (\$3))\,x_2 + (\$3)\,(x_1) - \$1 \\ &= (-(\$8)\,x_2 + (\$3))\,x_1 + (\$3)\,(x_2) - \$1. \end{aligned} \tag{1}$$

Therefore, if $x_1 = 3/8$ or $x_2 = 3/8$, then $E[W1] = \$1/8$, independent of the strategy of the other player. This means that Player #1 can maximize his winnings by randomly showing heads with probability 3/8, or Player #2 can minimize his losses by randomly showing heads with probability 3/8. Moreover, for $1/3 \le x_1 \le 2/5$ or $1/3 \le x_2 \le 2/5$, $E[W1] \ge \$0$. To see this, consider two cases. First, find the maximum value for $\varepsilon > 0$, such that $E[W1] \ge \$0$ subject to $x_1 = 3/8 + \varepsilon$. Substituting $x_1 = 3/8 + \varepsilon$ into the first expression in (1) leads to

$$E[W1] = -(\$8)\,\varepsilon x_2 + \left(\$\frac{1}{8}\right) + \$3\varepsilon \ge \$0. \tag{2}$$

From (2), the optimal strategy for Player #2 is $x_2 = 1$ (since Player #2 wants to minimize $E[W1]$), which leads to $\varepsilon \le 1/40$. Therefore, the maximum value for ε is 1/40, which leads to an upper bound of 2/5 for x_1. Similarly, the maximum value for $\delta > 0$, such that $E[W1] \ge \$0$ subject to $x_1 = 3/8 - \delta$ leads to $\delta \le 1/24$, resulting in a lower bound of 1/3 for x_1. Therefore, if Player #1 uses these strategies, then on average, Player #1 is guaranteed to win. A similar analysis results in the same bounds for x_2 such that $E[W1] \ge \$0$.

This paper presents a general analysis for a two player matching game first described in the newspaper column *Ask Marilyn* [1]. A generalization of this game to three players is also described and analyzed. The existence of equilibrium strategies can be established using known results based on Nash equilibrium [5]. However, the identification of the actual strategies to be used to obtain such equilibrium results cannot be obtained directly from the theory. Therefore, the results presented in this paper provide a practical illustration of how a game can be biased against a particular player provided that the remaining players work together (using carefully designed strategies) to achieve this objective. The results also illustrate that under certain circumstances, it is impossible for a game to be made unfair, no matter how well coordinated players act with each other against another player.

2. GENERALIZED MATCHING COIN GAME DESCRIPTION

To make the game more interesting, suppose that this matching game is generalized in terms of the winnings of the two players. In particular, suppose that Player #1 wins $\$U_1$ when HT is shown and $\$V_1$ when TH is shown. Similarly, Player #2 wins $\$U_2$ when HH is shown and $\$V_2$ when TT is shown. Then, the expected winnings for Player #1 is

$$\begin{aligned} E[W1] &= x_1\,(1-x_2)\,U_1 + (1-x_1)\,x_2V_1 - x_1x_2U_2 - (1-x_1)\,(1-x_2)\,V_2 \\ &= x_1\,(U_1+V_2) + x_2\,(V_1+V_2) - x_1x_2\,(U_1+U_2+V_1+V_2) - V_2 \\ &= x_1\,(U_1+V_2) + x_2\,[(V_1+V_2) - x_1\,(U_1+U_2+V_1+V_2)] - V_2. \end{aligned} \tag{3}$$

Proposition 1 provides the optimal value for x_1, such that $E[W1]$ is maximized.