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Sensitivity analysis scheme of boundary value problem of 2D Poisson equation by using Trefftz method

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Abstract

This paper describes the sensitivity analysis of the boundary value problem of two-dimensional Poisson equation by using Trefftz method. A non-homogeneous term of two-dimensional Poisson equation is approximated with a polynomial function in the Cartesian coordinates to derive a particular solution. The unknown function of the boundary value problem is approximate with the superposition of the T-complete functions of Laplace equation and the derived particular solution with unknown parameters. The parameters are determined so that the approximate solution satisfies boundary conditions. Since the T-complete functions and the particular solution are regular, direct differentiation of the expression results leads to the sensitivity expressions. The boundary-specified value and the shape parameter are taken as the variables of the sensitivity analysis to formulate the sensitivity analysis methods. The present scheme is applied to some numerical examples in order to confirm the validity of the present algorithm.

Keywords: Trefftz method; Computing point analysis scheme; T-complete function; Poisson equation; Sensitivity analysis

1. Introduction

The inverse problems are mathematically formulated as so-called optimization problems and therefore, it is very important to accurately estimate sensitivities which are the derivatives of an unknown function with respect to the design variables of the optimization problem. For this purpose, the sensitivity analysis schemes based on finite and boundary element methods have been presented by many researchers [1–16]. Finite element method (FEM) is a very powerful tool for analysis. However, when it is applied to the inverse problems such as the shape optimization problems, the successive shape modification of the object may distort or reverse the finite elements and therefore, the distortion may degrade the computational accuracy. For overcoming this difficulty, the use of boundary element method has been presented [4,5,8–10,12–16]. Boundary element method (BEM) can solve the problem by boundary discretization alone when the objects under consideration are governed with linear and homogeneous differential equations. Boundary integral equations, however, have singular property due to singular fundamental solutions and therefore, sensitivities are given by hyper-singular integral equations derived from differentiation of the original integral equation. Special formulation techniques, however, are necessary for deriving the integral equation from the hyper-singular differential equation. For overcoming the above difficulties, authors have presented the sensitivity analysis schemes derived from Trefftz method [17–19]. Trefftz method is a boundary-type solution formulated with regular T-complete functions satisfying governing differential equation [20–33]. The mathematical background of the Trefftz formulation is mainly discussed by Herrera [34], Qin [33] and so on. In the Trefftz method, the computational cost of the mesh generation is cheaper than FEM and moreover, the physical quantities and their sensitivities are represented with regular expressions. In the previous studies, authors have applied the sensitivity analysis schemes based on Trefftz method to two-dimensional potential problem [17,18], three-dimensional potential problem [35] and elastic problem [19]. Besides, in Ref. [36], the authors discussed the sensitivity analysis scheme based on the direct

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Trefftz formulation. On the other hand, the sensitivity analysis schemes for the boundary value problem of two-dimensional Poisson equation are presented in this paper.

When boundary-type solutions are applied to the boundary value problem of a non-homogeneous differential equation, a derived equation has a domain integral term derived from the non-homogeneous term of the governing equation. In this case, the domain integral term has to be discretized with finite elements. For overcoming this difficulty, the schemes to transform the domain integral term into the boundary integral term have been presented by some researchers; for example, Multiple Reciprocity Method (MRM) [37,38], Dual Reciprocity Method (DRM) [39] and computing point analysis scheme [40,41]. In the MRM, a domain integral term is transformed into the infinite series of boundary integral terms by the iterative application of the Gauss–Green formula. In the DRM, a non-homogeneous term is approximated by the relatively simple function and then, the domain integral term is transformed into the boundary integral term by using a particular solution related to the non-homogeneous term and applying the Gauss–Green formula. In the computing point analysis scheme, a non-homogeneous term including the unknown function is approximated by the fourth-order polynomial in the Cartesian coordinates and the domain integral term is also transformed into the boundary integral term by applying the Gauss–Green formula.

Authors have presented the following scheme for solving the boundary value problem of the two-dimensional Poisson equation in the previous study [42]. A non-homogeneous term of the governing equation is approximated with a polynomial function in Cartesian coordinates to derive the related particular functions. An unknown function is represented with the superposition of T-complete functions of the homogeneous differential equation and the particular solutions. Substituting the expression into the original boundary value problem results in the boundary value problem of the homogeneous differential equation. The derived boundary value problem is solved with the Trefftz method for the boundary value problem of the Laplace equation. Since the T-complete functions and the particular solutions are regular functions, direct differentiation of the expression leads to sensitivities. The specified values on the boundary and the shape parameters are taken as the design variable of the sensitivity analysis in order to explain the sensitivity analysis formulation. Finally, the present scheme is applied to the numerical examples in order to discuss the formulation. Trefftz formulation can be performed by means of the collocation, the least square or the Galerkin formulation. The collocation formulation is employed in this paper. From the view-point of the accuracy of the numerical solutions, the least square or the Galerkin formulation is better than the collocation method. The aim of this paper is to describe the sensitivity analysis formulation of the two-dimensional Poisson equation by using Trefftz method. Since the collocation formulation is easier than the other methods, the collocation method is more adequate in order to explain the present formulation for the sensitivity analysis. In Ref. [43], the present formulation has been applied to the sensitivity analysis with respect to the boundary-specified values. In this paper, the sensitivity analysis schemes with respect to both the boundary-specified value and the shape parameter are presented and the validity of the formulation are discussed on three numerical examples.

This paper is organized as follows. In Section 2, Trefftz formulation for Poisson equation is described. Sensitivity analysis schemes for a specified value on a boundary and a shape parameter are formulated in Sections 3 and 4, respectively. In Section 5, the present schemes are applied to some numerical examples. The numerical solutions by the present schemes are compared with theoretical and the numerical differential solutions. Some conclusions are summarized in Section 6.

2. Trefftz formulation for Poisson equation [42]

2.1. Trefftz formulation

Consider the governing equation and the boundary conditions of the boundary value problem of Poisson equation

$$\nabla^2 u + b(x, y, u, u_{,k}) = 0 \quad (\text{in } \Omega), \tag{1}$$

and

$$\left. \begin{aligned} u &= \bar{u}, \quad \text{on } \Gamma_u \\ q &= \bar{q}, \quad \text{on } \Gamma_q \end{aligned} \right\}, \tag{2}$$

where Ω, Γ_u and Γ_q denote the object domain and its potential and flux specified boundaries, respectively. The value $u_{,k}$ denotes the derivative of an unknown function u with respect to the variable k. Besides

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3}$$

Approximating the non-homogeneous term b with a polynomial function yields

$$b = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{r}. \tag{4}$$

In this study, the fifth-order polynomial is adopted for the function $\boldsymbol{r}$. Therefore, $\boldsymbol{c}$ and $\boldsymbol{r}$ are defined, respectively, as follows

$$\boldsymbol{c}^{\mathrm{T}} = \{c_1, c_2, \ldots, c_{21}\}, \tag{5}$$

$$\begin{aligned} \boldsymbol{r}^{\mathrm{T}} &= \{r_1, r_2, r_3, \ldots, r_{21}\} \\ &= \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, \\ &\quad xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\}. \end{aligned} \tag{6}$$