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Shape sensitivity analysis and the energy momentum tensor for the kinematic and static models of torsion

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Abstract

The aim of this paper is to bridge shape sensitivity analysis and configurational mechanics by means of a widespread use of the shape derivative concept. This technique will be applied as a systematic procedure to obtain the Eshelby's energy momentum tensor associated to the problem under consideration. In order to highlight special features of this procedure and without loss of generality, we focus our attention in the application of shape sensitivity analysis to the problem of twisted straight bars within the framework of linear elasticity.

Kinematic and static variational formulations as well as the direct method of sensitivity analysis are used to perform shape derivatives of both models. Integral expressions of first and second order shape derivatives of the total potential energy and the complementary potential energy with respect to an arbitrary transverse cross-section shape change, are achieved. These integral expressions put in evidence the relationship between shape sensitivity analysis and the first and second order Eshelby's energy momentum tensors. Also, the null divergence property of these tensors is easily proved by comparing, in each case, the domain and boundary integral shape derivative arrived at. Finally, an example with a known exact solution, corresponding to an elastic bar with elliptical transverse cross-section submitted to twist, is presented in order to illustrate the usefulness of these tensors to compute the corresponding shape derivatives.

Keywords: Shape sensitivity analysis; Direct method; Energy momentum tensor; Twisted elastic bars; Variational formulation

1. Introduction

Sensitivity analysis commenced as a branch of structural optimization. Strongly linked since their origin, both disciplines have developed and achieved together a high degree of sophistication and success in several applications. Recently, researchers have focused their attention on the problem in which the shape, or more specifically the domain over which the problem is defined, becomes the design variable. This problem, known in literature as *shape sensitivity analysis*, has received special attention over the past years when its

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mathematical foundations have been established. Proposed originally by Cèa (1981), has been largely developed by Zolésio (1981), Rousselet (1987) and Masmoudi (1987), and is widely discussed in the book of Haug, Choi and Komkov. See also van Keulen et al. (2005), where 239 references on sensitivity analysis are included.

On the other hand, seemingly disconnected from shape sensitivity analysis, *configurational mechanics* is a branch of continuum mechanics that has undergone processes in which simultaneous deformation and configurational change of a body take place (Herrmann and Kienzler, 2001; Gurtin, 2000). Also referred to as Eshelbian Mechanics, which honors Eshelby's successful work associating the concept of configurational force on a material defect with the change of the total energy of the system with respect to a possible displacement of such defect. Even since the renowned contribution of Eshelby came to light, several researches have focused their attention in this field, which has recently grown due to its vast range of applications. As prime examples, one might mention crack advance, cavitation, void nucleation and growth, change in the geometry of the free boundary, as well as the motion of an interface between two phases of a material.

As stated by the authors in previous papers, in the particular case of fracture mechanics, if crack advance is simulated as a shape change, the expression of the energy release rate can be obtained through the shape derivative of the total potential energy (Taroco, 2000). Also shape sensitivity analysis, together with numerical methods, such as the finite element method, combined with post processing techniques, can provide reliable numerical results of the energy release rate of 2D and 3D cracked bodies (Feijóo et al., 2000). To improve the accuracy of the numerical values of the fracture parameter an error estimator including re-meshing procedures is reported in (Saliba et al., 2005). In other situations when it is more favorable for the material to dissipate energy opening up cavities, the topological-shape sensitivity analysis can be used to perform the sensitivity of the cost function when a cavity is created (Novotny et al., 2003). For elastic shells within the framework of Reissner's theory, the corresponding Eshelby's tensor is linked to the shape derivative of the total potential energy with respect to changes of curvature and size of the shell (Taroco and Feijóo, 2004).

However, shape sensitivity analysis can be viewed as a well developed area of research, with not all of its inherent aspects related to configurational mechanics deeply explored yet. In particular, the shape sensitivity analysis could be expressed through boundary integrals. In doing so, some important information is loss. In fact it is not difficult to demonstrate that shape derivatives (of any order) could be written by domain or boundary integrals where two elements can be clearly identified. The first one is related with the velocity field which defines the shape change. The other one could be identified as what we denoted as Eshelby's tensors.

From previous considerations the present paper addresses shape sensitivity analysis as a systematic procedure to obtain the Eshelby's tensors of any order associated to the problem under consideration. Thus, we have elected to explore the derivation of the first and second order shape sensitivity of a twisted elastic straight bar with uniform transverse cross-section. By doing so, our goal is to highlight certain special features that are clearly distinguishable in this particular case. Under the previous conditions, the analysis tends to be simpler than other cases and, at the same time, does not lose generality in the obtained results. In fact, the most popular approaches in solid mechanics, the kinematic and the static models, can be easily stated for torsion problems.

As known, the theory of bars under twist conditions, can be formulated as a bi-dimensional problem by means of simplified assumptions (Sokolnikoff, 1974). The Saint-Venant's kinematic approach and Prandtl's static approach, should be used. Both formulations, assume the external action of twist induces in the bar a pure shearing strain–stress state, leading to the well known Laplace's type equation with Neumann and Dirichlet boundary conditions, respectively. The unknowns are scalar fields, the rate of twist and the warping of the transverse cross-section for the Saint-Venant's approach and the stress function for the Prandtl's approach. Moreover and due to its application in several engineering problems, researchers from different fields have studied Laplace's equation extensively. For this reason, physical, mathematical, numerical and computational information for this equation abounds.

Within the aim of providing a clear presentation and, at the same time, in order to simplify the shape sensitivity analysis, we adopt as cost function the potential of the state equation, using the terminology of shape optimization. In other words, we use the potential strain energy in the case of the kinematic model and the complementary potential energy to the static model. Closed expressions of the shape derivatives of both cost functions as domain or boundary integrals may therefore be obtained in terms of the strain–stress state of the bar and the adopted shape change velocity (Buscaglia et al., 1997).