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Analytical sensitivity analysis of geometrically nonlinear structures based on the co-rotational finite element method

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Abstract

This paper is concerned with the parameter sensitivity analysis of structures undergoing large displacements. The authors introduce the analytical sensitivity expressions for an element independent co-rotational formulation of a geometrically nonlinear finite element method. An extension of this formulation to treat follower forces is presented. The co-rotational framework uses a pre-existing linear finite element library and does not require the development and implementation of kinematically nonlinear element formulations. This feature along with the element independence makes the co-rotational framework an attractive option for the implementation of geometrically nonlinear analysis and sensitivity analysis capabilities. The sensitivity formulations with respect to shape and material parameters are presented and their numerical treatment is discussed. The importance of a consistent tangent stiffness, including unsymmetric terms, on the accuracy of computed sensitivities is addressed. The framework is applied to shape and topology optimization examples, verifying the methodology and highlighting the importance of accounting for large displacement effects in design optimization problems.

Keywords: Co-rotational formulation; Sensitivity analysis; Shape optimization; Topology optimization; Geometrically nonlinear; Follower force

1. Introduction

The analysis and design of structures is often dominated by geometrically and materially nonlinear effects. While the structural integrity needs to be checked against buckling and yielding, accounting for large displacements is required to guarantee proper functionality of the structure. The latter is in particular important for the design of so-called compliant mechanisms and actuators, which undergo large elastic displacements. Such devices have recently gained substantial popularity in the design of smart adaptive structures [1] and in the microelectromechanical systems (MEMS) community [2,3].

Formal design optimization methods are an appealing tool for the design of such structures as high-fidelity numerical simulation methods, such as nonlinear finite element analysis, can be integrated into the design process in a systematic fashion [4]. In the context of mechanism design for example, Buhl et al. [5] and Gea and Luo [6] have included geometric nonlinearities in the topological design of minimum compliance structures. Bruns and Tortorelli [7] and Pedersen et al. [8] have optimized large displacement compliant mechanisms with continuum finite elements, while Saxena and Ananthasuresh [9] have done the same using beam elements.

A broad class of engineering design problems can be efficiently solved by gradient-based optimization algorithms. These algorithms call for the evaluation of the design criteria, which are the objective and constraints, as well as their sensitivities with respect to the optimization variables that describe the geometrical and/or material properties of the structure. Design criteria frequently used to characterize the mechanical behavior of a structure for design purposes include displacements, stresses, and strain energy, where in general the criteria depend on the structural response. While the evaluation of the design criteria requires most often no or only minor additions to existing analysis software, the computation of the sensitivities may be a task that is far more involved, depending on the type of sensitivity analysis used.

Sensitivity analysis can be subdivided into analytical methods and numerical approaches. Most popular among numerical

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approaches are finite differencing schemes that use existing analysis software in a "black-box" mode but are afflicted with large computational costs. Furthermore, the accuracy of the results depends on the proper choice for the size of the perturbation factor, which is most often unknown. The numerical effort associated with finite differencing methods increases linearly with the number of optimization variables, making finite difference schemes impractical for problems that include a large number of optimization variables, a situation commonly encountered in topology optimization for example. A numerical approach that is less sensitive to the perturbation size is the complex variable method [10]. This method, however, requires access to the source code and doubles the computational costs. Analytical methods overcome the computational and accuracy shortcomings of numerical approaches but require the formulation and solution of the so-called sensitivity equations.

The complexity of analytically computing the sensitivities is illustrated by the following generic example that assumes a finite element discretization of the governing equations. Let q_j be a criterion and s_i an optimization variable; the derivative $\mathrm{d}q_j/\mathrm{d}s_i$ can be written as follows:

$$\frac{\mathrm{d}q_j}{\mathrm{d}s_i} = \frac{\partial q_j}{\partial s_i} + \frac{\partial q_j}{\partial \mathbf{u}}^{\mathrm{T}} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s_i}, \tag{1}$$

where $\mathbf{u}$ denotes the displacement vector and the superscript T is the transpose of a matrix or vector. The derivatives of the displacements can be evaluated by solving the differentiated governing equations. The nonlinear static equilibrium equations can be written in discretized form as follows:

$$\mathbf{R}(\mathbf{u}) = \mathbf{f}^{\mathrm{int}}(\mathbf{u}) - \mathbf{f}^{\mathrm{ext}}(\mathbf{u}) = \mathbf{0}, \tag{2}$$

where $\mathbf{f}^{\mathrm{int}}$ and $\mathbf{f}^{\mathrm{ext}}$ denote the internal and external forces, respectively. Differentiating the above equation at equilibrium yields

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s_i} = 0; \quad \mathbf{K}_{\mathrm{T}} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s_i} = -\frac{\partial \mathbf{R}}{\partial s_i}, \tag{3}$$

where $\mathbf{K}_{\mathrm{T}}$ is the Jacobian of the structural residual evaluated at equilibrium. Replacing the derivatives of the displacements in (1) by the symbolic solution of (3) yields

$$\frac{\mathrm{d}q_j}{\mathrm{d}s_i} = \frac{\partial q_j}{\partial s_i} - \frac{\partial q_j^{\mathrm{T}}}{\partial \mathbf{u}} \mathbf{K}_{\mathrm{T}}^{-1} \frac{\partial \mathbf{R}}{\partial s_i}, \tag{4}$$

where the superscript -1 denotes the matrix inverse. The second term on the right-hand side can be evaluated either by the direct method or the adjoint method depending on the number of design criteria compared to the number of optimization variables. In both cases, terms containing partial derivatives with respect to the displacements and optimization variables need to be evaluated. In particular, the partial derivative of the residual

$$\frac{\partial \mathbf{R}}{\partial s_i} = \frac{\partial \mathbf{f}^{\mathrm{int}}}{\partial s_i} \frac{\partial \mathbf{f}^{\mathrm{ext}}}{\partial s_i} \tag{5}$$

requires the differentiation of the finite element formulations. For mature finite element codes with a large element library this leads to the huge undertaking of differentiating and implementing the partial derivatives. This burden can be attenuated by using either automatic differentiation tools, such as ADIFOR [11], or by approximating the partial derivative by finite differences. The latter approach is often referred to as semi-analytical sensitivity analysis and, if no additional measures are provided, may lead to significant approximation errors, in particular for slender structures [12,13].

Most of the published works on analytical sensitivity analysis of geometrically nonlinear structures employ either a total or an updated Lagrangian formulation for the nonlinear finite element description [4–9,14]. This requires the implementation and differentiation of multiple kinematically nonlinear element formulations in order to create a library of geometrically nonlinear elements. In contrast to the workload introduced by a total or updated Lagrangian formulation, the co-rotational formulation of geometrically nonlinear finite element analysis offers a reduction in the implementation costs of introducing geometrically nonlinear analysis and the associated sensitivity equations.

The co-rotational framework is founded on the concept of satisfying the equilibrium equations in a reference system that is local to each element and moves rigidly along with the deformed element [15–17]. The motion of this reference system defines the rigid body motion of the element undergoing large displacements, allowing for a conventional small strain element from linear analysis to model the elastic strains arising from the deformation relative to the local system. Work by Crisfield [18] and Nour-Omid and Rankin [19] have addressed the issue of consistency in the stiffness definitions, which can be complicated by the variation of finite rotations [20]. So far, the application of analytical sensitivity analysis methods to co-rotational finite elements has been limited to bar and beam elements [21–23]. However, the element independent nature of the formulation easily lends itself to generalized derivations of the sensitivity expressions. These general expressions are introduced in this paper.

Utilization of the co-rotational finite element formulation will be shown to limit the implementation efforts associated with analytical sensitivity methods, specifically the derivation of structural sensitivities with respect to both geometrical and material parameters of geometrically nonlinear structures. The creation of a geometrically nonlinear element from an already existing linear finite element, which presumably has a significant amount of existing analytical sensitivities, provides for the reuse of many element dependent routines. The element independent framework of the formulation allows for relative ease in the implementation of new geometrically nonlinear elements, including differentiation with respect to structural parameters. Additionally, certain follower forces, such as thermal or pressure loads, are elegantly defined due to the use a linear element as a conceptual base. Although the co-rotational formulation is limited in application to small strain problems due to the linear element at its core, this model is sufficient for numerous engineering applications, most notably mechanism design.