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Sensitivity analysis using probability bounding

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Abstract

Probability bounds analysis (PBA) provides analysts a convenient means to characterize the neighborhood of possible results that would be obtained from plausible alternative inputs in probabilistic calculations. We show the relationship between PBA and the methods of interval analysis and probabilistic uncertainty analysis from which it is jointly derived, and indicate how the method can be used to assess the quality of probabilistic models such as those developed in Monte Carlo simulations for risk analyses. We also illustrate how a sensitivity analysis can be conducted within a PBA by pinching inputs to precise distributions or real values.

Keywords: Probability bounds analysis; Interval analysis; Second-order probability; Sensitivity analysis; Convolution; Robust Bayes; Bayesian sensitivity analysis

1. Introduction

Uncertainty analysis is a systematic study in which "a neighborhood of alternative assumptions is selected and the corresponding interval of inferences is identified" [1]. There are two disparate ways to effect such a study. One natural way is to bound the neighborhood with interval ranges. Another natural way is to ascribe a probability distribution to the elements in the neighborhood. Consider, for example, the context of a deterministic calculation. When the model involves uncertainty about the real-valued quantities used in the calculation, uncertainty analysis can be conducted via interval analysis [2–5]. Probability theory, implemented perhaps by Monte Carlo simulation, can also be used as an uncertainty analysis of a deterministic calculation because it yields a distribution describing the probability of alternative possible values about a point estimate [6–9]. In the figure below these two possible paths are shown as right and left downward arrows, respectively (Fig. 1).

Of course, the calculations on which it might be desirable to conduct uncertainty analyses are not all deterministic. In fact, many of them are already probabilistic, as is the case in most modern risk analyses and safety assessments. One could construct a probabilistic uncertainty analysis of a probabilistic calculation. The resulting analysis would be a second-order probabilistic assessment. However, such studies can be difficult to conduct because of the large number of calculations that are required. It is also sometimes difficult to visualize the results in a way that is easily comprehensible. Alternatively, one could apply bounding arguments to the probabilistic calculation and arrive at interval versions of probability distributions. We call such calculations "probability bounds analysis" (PBA) [10–12]. This approach represents the uncertainty about a probability distribution by the set of cumulative distribution functions lying entirely within a pair of bounding distribution functions called a "probability box" or a "p-box". (The mathematical definition of a p-box is given in a companion paper [13] in this journal issue.) PBA is an uncertainty analysis of a probabilistic calculation because it defines neighborhoods of probability distributions (i.e. the p-boxes) that represent the uncertainty about imperfectly known input distributions and projects this uncertainty through the model to identify a neighborhood of answers (also characterized by a p-box) in a way that guarantees the resulting bounds will entirely enclose the cumulative distribution function of the output. A probability distribution is to a p-box the way a real scalar number is to an interval. The bounding distributions of the p-box

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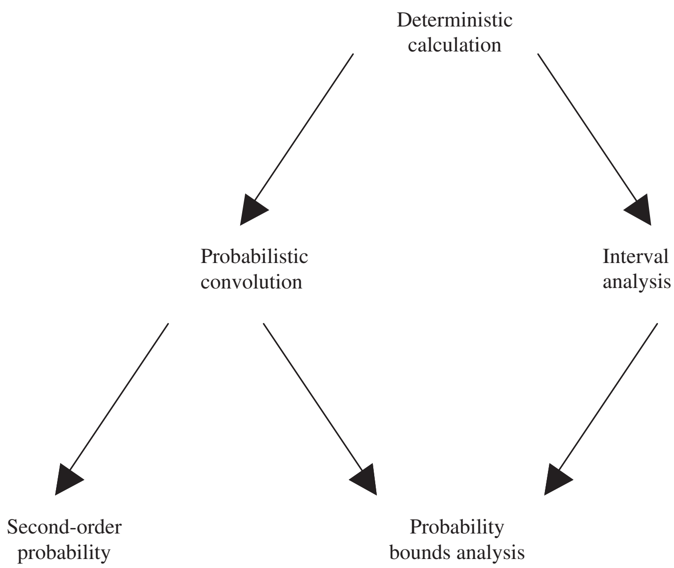

Fig. 1. Relationships among different calculation strategies. Arrows represent generalizations.

enclose all possible distributions in the same way that the endpoints of the interval circumscribe the possible real values.

PBA is related to other forms of uncertainty analysis. It is a marriage of probability theory and interval analysis. As depicted in Fig. 1, PBA can arise either by bounding probability distributions (the left path down to PBA) or by forming probability distributions of intervals (the right path). PBA is not simply an interval analysis with probability distributions. It is an integration of the two approaches that generalizes and is faithful to both traditions. For instance, when PBA is provided the same information as is used in a traditional Monte Carlo assessment (i.e. precise information about input distributions and their interdependencies), PBA will yield the same answers as the Monte Carlo simulation. When provided only range information about the inputs, PBA will yield the same answers as an interval analysis.

PBA permits a comprehensive uncertainty analysis that is an alternative to complicated second-order or nested Monte Carlo methods. PBA is very similar in spirit to Bayesian sensitivity analysis (which is also known as robust Bayes [14]), although the former exclusively concerns arithmetic and convolutions, and the latter often addresses the issues of updating and aggregation. Unlike Bayesian sensitivity analysis, PBA is always easy to employ on problems common in risk analyses of small and moderate size because it does not depend on the use of conjugate pairs to make calculations simple. PBA is a practical approach to computing with imprecise probabilities [15]. Like a Bayesian sensitivity analysis, imprecise probabilities are represented by a class of distribution functions. PBA is simpler because it defines the class solely by reference to two bounding distributions. It therefore cannot fully represent a situation in which there are intermediate distributions lying within the bounds that are excluded from the class. Indeed, p-boxes will often contain distributions that, if isolated and presented to an expert, would be rejected as quite far-fetched. However, in contexts of risk and safety assessments, this may not be a significant drawback if the analyst is principally concerned with the tail risks governing the probability of extreme events and not so much with the shapes of the distributions being enveloped.

Because PBA is a marriage of probability theory and interval analysis, it treats variability (aleatory uncertainty) and incertitude (epistemic uncertainty) separately and propagates them differently so that each maintains its own character. The distinction between these two forms of uncertainty is considered very important in practical risk assessments [16]. PBA is useful because it can account for the distinction when analysts think it is important, but the method does not require the distinction in order to work. The two forms of uncertainty are like ice and snow in that they often seem to be very different, but, when studied closely, they can sometimes become harder and harder to distinguish from each other. An advantage of PBA, and imprecise probability methods generally [15], is that they can be developed in behavioral terms that do not depend on maintaining a strict distinction between the two forms of uncertainty which can be problematic.

2. PBA circumscribes possible distributions given uncertainty

PBA can produce rigorous bounds around the output distribution from an assessment. These bounds enclose all the possible distributions that could actually arise given what is known and what is not known about the model and its inputs. Because it is based on the idea of bounding rather than approximation, it provides an estimate of its own reliability [17,18, cf. 19]. PBA can comprehensively account for possible deviations in assessment results arising from uncertainty about

- distribution parameters,
- distribution shape or family,
- intervariable dependence, and even
- model structure.

Moreover, it can handle all of these kinds of uncertainties in a single calculation that gives a simple and rigorous characterization of how different the result could be given all of the professed uncertainty. The requisite computations used in PBA are actually quite simple and have been implemented in straightforward algorithms [11,17,18,20–22]. Several of the basic formulas are reviewed in the companion paper [13]. For a large class of problems of moderate size commonly encountered in risk analysis,