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Sensitivity analysis in conjunction with evidence theory representations of epistemic uncertainty

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Abstract

Three applications of sampling-based sensitivity analysis in conjunction with evidence theory representations for epistemic uncertainty in model inputs are described: (i) an initial exploratory analysis to assess model behavior and provide insights for additional analysis; (ii) a stepwise analysis showing the incremental effects of uncertain variables on complementary cumulative belief functions and complementary cumulative plausibility functions; and (iii) a summary analysis showing a spectrum of variance-based sensitivity analysis results that derive from probability spaces that are consistent with the evidence space under consideration.

Keywords: Epistemic uncertainty; Evidence theory; Sensitivity analysis; Uncertainty analysis; Variance decomposition

1. Introduction

Uncertainty analysis and sensitivity analysis should be important components of any analysis of a complex system, with (i) uncertainty analysis providing a representation of the uncertainty present in the estimates of analysis outcomes and (ii) sensitivity analysis identifying the contributions of individual analysis inputs to the uncertainty in analysis outcomes [1]. Probability theory provides the mathematical structure traditionally used in the representation of epistemic (i.e., state of knowledge) uncertainty, with the uncertainty in analysis outcomes represented with probability distributions and typically summarized as cumulative distribution functions (CDFs) or complementary cumulative distribution functions (CCDFs) [2–4]. A variety of sensitivity analysis procedures have been developed for use in conjunction with probabilistic representations of uncertainty, including differential analysis [5,6], the Fourier amplitude sensitivity test (FAST) and related variance decomposition procedures [7–11], regression-based techniques [12,13], and searches for nonrandom patterns [14]. Additional background information on uncertainty and sensitivity analysis is available in several review presentations [1,14–19].

Although probabilistic representations of uncertainty have been successfully employed in many analyses, such representations have been criticized for inducing an appearance of more refined knowledge with respect to the existing uncertainty than is really present [20,21]. Much of this criticism derives from the use of uniform distributions to characterize uncertainty in the presence of little or no knowledge with respect to where the appropriate value to use for a parameter is located within a set of possible values [22, pp. 52–62]. As a result, a number of alternative mathematical structures for the representation of epistemic uncertainty have been proposed, including evidence theory, possibility theory, and fuzzy set theory [23–26].

Evidence theory provides a promising alternative to probability theory that allows for a fuller representation of the implications of uncertainty than is the case in a probabilistic representation of uncertainty [27–33]. In particular, evidence theory involves two representations of the uncertainty associated with a set of possible analysis

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inputs or results: (i) a belief, which provides a measure of the extent to which the available information implies that the true value is contained in the set under consideration, and (ii) a plausibility, which provides a measure of the extent to which the available information implies that the true value might be contained in the set under consideration. One interpretation of the belief and plausibility associated with a set is that (i) the belief is the smallest possible probability for the set that is consistent with all available information and (ii) the plausibility is the largest possible probability for the set that is consistent with all available information. An alternative interpretation is that evidence theory is an internally consistent mathematical structure for the representation of uncertainty without any explicit conceptual link to probability theory. The mathematical operations associated with evidence theory are the same for both interpretations. Just as probability theory uses CDFs and CCDFs to summarize uncertainty, evidence theory uses cumulative belief functions (CBFs), cumulative plausibility functions (CPFs), complementary cumulative belief functions (CCBFs), and complementary cumulative plausibility functions (CCPFs) to summarize uncertainty.

Although evidence theory is beginning to be used in the representation of uncertainty in applied analyses, the authors are unaware of any attempts to develop sensitivity analysis procedures for use in conjunction with evidence theory. Due to the importance of sensitivity analysis in any decision-aiding analysis, the potential usefulness of evidence theory will be enhanced if meaningful and practicable sensitivity analysis procedures are available for use in analyses that employ evidence theory in the representation of uncertainty. As a result, the focus of this presentation is on the development of sensitivity analysis procedures for use in conjunction with evidence theory representations of uncertainty.

After a brief overview of evidence theory (Section 2), the following topics are considered: (i) exploratory sensitivity analysis (Section 3), (ii) use of sensitivity analysis results in the stepwise construction of CCBFs and CCPFs (Section 4), (iii) summary sensitivity analysis of evidence theory representations of uncertainty (Section 5), (iv) example results (Section 6), (v) formal justification of procedure used in stepwise construction of CCBFs and CCPFs (Section 7), and (vi) concluding summary (Section 8).

2. Evidence theory

Evidence theory is based on the specification of a triple $(\mathcal{S}, \mathbb{S}, m)$, where (i) $\mathcal{S}$ is the set that contains everything that could occur in the particular universe under consideration, (ii) $\mathbb{S}$ is a countable collection of subsets of $\mathcal{S}$, and (iii) m is a function defined on subsets of $\mathcal{S}$ such that $m(\mathcal{E})>0$ if $\mathcal{E} \in \mathbb{S}$, $m(\mathcal{E})=0$ if $\mathcal{E} \subset \mathcal{S}$ and $\mathcal{S} \notin \mathbb{S}$, and $\Sigma_{\mathcal{E}\in\mathbb{S}} m(\mathcal{E})=1$. For a subset $\mathcal{E}$ of $\mathcal{S}$, $m(\mathcal{E})$ characterizes the amount of "likelihood" that can be assigned to $\mathcal{E}$ but to no proper subset of $\mathcal{E}$. In the terminology of evidence theory, (i) $\mathcal{S}$ is the sample space or universal set, (ii) $\mathbb{S}$ is the set of focal elements for $\mathcal{S}$ and m, and (iii) $m(\mathcal{E})$ is the basic probability assignment (BPA) associated with a subset $\mathcal{E}$ of $\mathcal{S}$. The elements of $\mathcal{S}$ are often vectors $\mathbf{x}=[x_1, x_2, \ldots, x_n]$, where each element x_i of $\mathbf{x}$ is a variable with its own evidence space $(\mathcal{S}_i, \mathbb{S}_i, m_i)$. When the x_i's are assumed to be independent, (i) $m(\mathcal{E})=\Pi_i m_i(\mathcal{E}_i)$ if $\mathcal{E}=\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ and $\mathcal{E}_i \in \mathbb{S}_i$ for $i=1,2,\ldots,n$, and (ii) $m(\mathcal{E})=0$ otherwise. An evidence space $(\mathcal{S}, \mathbb{S}, m)$ plays the same role in evidence theory that a probability space $(\mathcal{P}, \mathbb{P}, p)$ plays in probability theory, where $\mathcal{P}$ is the sample space, $\mathbb{P}$ is a suitably restricted set of subsets of $\mathcal{P}$ (i.e., a σ-algebra), and p is the function (i.e., probability measure) that assigns probabilities to elements of $\mathbb{P}$ [34, Section IV.3].

The belief, $\mathrm{Bel}(\mathcal{E})$, and plausibility, $\mathrm{Pl}(\mathcal{E})$, for a subset $\mathcal{E}$ of $\mathcal{S}$ are defined by

$$\mathrm{Bel}(\mathcal{E}) = \sum_{\mathcal{U}\subset\mathcal{E}} m(\mathcal{U}) \quad \text{and} \quad \mathrm{Pl}(\mathcal{E}) = \sum_{\mathcal{U}\cap\mathcal{E}\neq\phi} m(\mathcal{U}). \tag{2.1}$$

In concept, $\mathrm{Bel}(\mathcal{E})$ is the amount of "likelihood" that must be assigned to $\mathcal{E}$, and $\mathrm{Pl}(\mathcal{E})$ is the maximum amount of "likelihood" that could possibly be assigned to $\mathcal{E}$. When the elements of $\mathcal{S}$ are real valued, a CCBF and a CCPF provide a convenient summary of an evidence space $(\mathcal{S}, \mathbb{S}, m)$ and correspond to plots of the points

$$\mathcal{CCBF} = \{[v, \mathrm{Bel}(\mathcal{S}_v)], v \in \mathcal{S}\} \quad \text{and}$$
$$\mathcal{CCPF} = \{[v, \mathrm{Pl}(\mathcal{S}_v)], v \in \mathcal{S}\}, \tag{2.2}$$

where $\mathcal{S}_v = \{x : x \in \mathcal{S} \text{ and } x>v\}$.

An important situation in the application of evidence theory is the consideration of a variable $y=f(\mathbf{x})$, where f is a function defined for elements $\mathbf{x}$ of the sample space $\mathcal{X}$ associated with an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ and $\mathbf{x}$ is represented as a vector because this is the case in most real analyses. The properties of f and $(\mathcal{X}, \mathbb{X}, m_X)$ induce an evidence space $(\mathcal{Y}, \mathbb{X}, m_Y)$ on y, which provides a characterization of the uncertainty associated with y. In turn, this uncertainty can be summarized with a CCBF and a CCPF defined by

$$\mathcal{CCBF} = \{[v, \mathrm{Bel}_X\{f^{-1}(\mathcal{Y}_v)\}], v \in \mathcal{Y}\} \quad \text{and}$$
$$\mathcal{CCBF} = \{[v, \mathrm{Pl}_X\{f^{-1}(\mathcal{Y}_v)\}], v \in \mathcal{Y}\}, \tag{2.3}$$

where Bel_X and Pl_X denote belief and plausibility defined with respect to $(\mathcal{X}, \mathbb{X}, m_X)$ and $\mathcal{Y}_v = \{y : y \in \mathcal{Y} \text{ and } y>v\}$. The generation and analysis of CCBFs and CCPFs of the preceding form are fundamental parts of the use of evidence theory to characterize the uncertainty in model predictions.

Additional discussion of evidence theory and its relationship to probability theory employing the same notation used in this presentation is available elsewhere [24,35]. Evidence theory derives from initiating work by Dempster [27,36,37] and Shafer [28], and as a result, Dempster–Shafer theory is often used as an alternative designation for evidence theory. Evidence theory has been widely studied and a number of summary references are available (e.g., [29–33]).