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Open Boundary Control Problem for Navier-Stokes Equations Including a Free Surface: Adjoint Sensitivity Analysis

I. YU. GEJADZE* AND G. J. M. COPELAND
Department of Civil Engineering
University of Strathclyde
John Anderson Building, 107 Rottenrow
Glasgow, G4 0NG, U.K.
`<igor.gejadze><g.m.copeland>@strath.ac.uk`

Abstract—This paper develops the adjoint sensitivities to the free-surface barotropic Navier-Stokes equations in order to allow for the assimilation of measurements of currents and free-surface elevations into an unsteady flow solution by open-boundary control. To calculate a variation in a surface variable, a mapping is used in the vertical to shift the problem into a fixed domain. A variation is evaluated in the transformed space from the Jacobian matrix of the mapping. This variation is then mapped back into the original space where it completes a tangent linear model. The adjoint equations are derived using the scalar product formulas redefined for a domain with variable bounds. The method is demonstrated by application to an unsteady fluid flow in a one-dimensional open channel in which horizontal and vertical components of velocity are represented as well as the elevation of the free surface (a 2D vertical section model). This requires the proper treatment of open boundaries in both the forward and adjoint models. A particular application is to the construction of a fully three-dimensional coastal ocean model that allows assimilation of tidal elevation and current data. However, the results are general and can be applied in a wider context.

Keywords—Navier-Stokes equations, Free surface, Open boundary, Optimal control, Adjoint equations, Sensitivity analysis, Ocean, Waves.

1. INTRODUCTION

The simulation of water circulation in coastal areas requires the application of either two-dimensional (2D) or three-dimensional (3D) computer flow models. These models calculate

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*Author to whom all correspondence should be addressed.

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solutions for either the 2D shallow-water equations (SWE) [1], or for the layer-averaged 3D Navier-Stokes equations (NSE) [2], or for the 3D free-surface Navier-Stokes formulation (fsNSE) using sigma coordinates [3], or for the fully 3D nonhydrostatic fsNSE [4,5], etc. All of these calculations require that the solution is driven by an unsteady inflow Dirichlet boundary condition based on 'known' data. However, in a typical application it is unlikely that measurements are available at the boundary but it is much more likely that data is available at scattered locations within the model domain, measured by current meters and tide gauges, for example. Hence the boundary conditions are actually unknown and must be recovered by a process of adjustments until the model solution agrees with measured data at the internal points. This describes the inverse problem in which data is assimilated into a model solution and boundary conditions are recovered from internal data. The process of adjustment can be systematized by calculating appropriate sensitivities to guide a gradient descend algorithm. In the field of meteorology, the data-assimilation techniques based on optimal control methods appear in early 1970s [6,7]. A general sensitivity theory for nonlinear systems was formulated in [8]. Boundary-control problems for free-surface fluid flow were considered for the SWE [9,10], for a depth-integrated tidal model [11], and for the vorticity equation [12,13], etc. For the primitive hydrostatic equation (PHE) we can refer to [14], where the authors declare a *discrete* adjoint fsNSE option. The *discrete* (consistent) adjoint refers to the adjoint of the discretized model equations. It is quite likely that similar *discrete* adjoint codes have been generated for some other ocean models. We present here a *continuous* (inconsistent) adjoint model, which is an important tool for general qualitative analysis, for considering solvability issues, etc. Also, of course, it is a valid way to construct the adjoint solver for practical applications. This paper presents the adjoint formulation of the nonhydrostatic barotropic fsNSE in 2D vertical section (for the inviscid case this formulation first appeared in [15]). The method is general and can be extended to the 3D baroclinic fsNSE. The novelty of this work is in the complete treatment of the free surface in the adjoint problem and in the clarification of open-boundary conditions in both the forward and adjoint models.

Tidal flows in shallow coastal waters are often represented by solutions to the SWE. This is usually considered to be valid for well-mixed, barotropic conditions. It is clear that in cases where freshwater inflows cause baroclinic conditions a 3D representation is needed which must also include the salinity transport equations. However, 3D models offer advantages even for barotropic conditions because the effects of topographic steering are better represented. For example, flows over or around a submerged step or shoal may cause flow separation and enhanced mixing that can only be represented in 3D. Even flow past a change in coastline direction may result in different flows at the surface where there may be a tendency to separate, than near to the bed where the flow remains attached. The resulting lateral shears cause an increase in mixing that may be significant in a pollutant transport study.

The rapid improvement in reliability and availability of data from coastal waters is driving the need for data-assimilation methods effective in tidally dominated flows. Adjoint models developed for oceanographic (deep water) applications, see for example [16], do not determine sensitivities to variations in the free surface and therefore cannot directly assimilate water-elevation data (although it can be achieved by introducing the measured geopotential surface). The model presented here allows direct assimilation of unsteady water levels by the adjoint method applicable to shallow tidal flows (as well as to deep water flows). This is probably the first fsNSE *continuous* adjoint model that is free from the hydrostatic assumption in any form. Further applications for the method are to the creation of operational coastal flow models that assimilate measurements of current flows and water levels in order to calculate a 'nowcast' of flow conditions at all locations within the model area. This in turn can be used as an initial condition for a short-term forecast.