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Second order topological sensitivity analysis

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Abstract

The topological derivative provides the sensitivity of a given cost function with respect to the insertion of a hole at an arbitrary point of the domain. Classically, this derivative comes from the second term of the topological asymptotic expansion, dealing only with infinitesimal holes. However, for practical applications, we need to insert holes of finite size. Therefore, we consider one more term in the expansion which is defined as the second order topological derivative. In order to present these ideas, in this work we apply the topological-shape sensitivity method as a systematic approach to calculate first as well as second order topological derivative for the Poisson's equations, taking the total potential energy as cost function and the state equation as constraint. Furthermore, we also study the effects of different boundary conditions on the hole: Neumann and Dirichlet (both homogeneous). Finally, we present some numerical experiments showing the influence of the second order topological derivative in the topological asymptotic expansion, which has two main features: it allows us to deal with hole of finite size and provides a better descent direction in optimization process.

Keywords: Topological asymptotic expansion; First order topological derivative; Second order topological derivative

1. Introduction

The topological derivative provides the sensitivity of a given cost function with respect to the insertion of an infinitesimal hole at an arbitrary point of the domain (Céa et al., 2000; Eschenauer et al., 1994; Novotny et al., 2003; Sokolowski and Żochowski, 1999). This derivative has been used as a descent direction to solve several problems, among others: topology optimization and inverse problems (Amstutz, 2005; Amstutz et al., 2005; Eschenauer and Olhoff, 2001; Feijóo et al., 2003, in press; Garreau et al., 2001; Lewinski and Sokolowski, 2003; Novotny et al., 2005; Samet et al., 2003). Classically, the topological derivative comes from the second term of the topological asymptotic expansion, dealing only with infinitesimal holes. However, for practical applications, we need to insert holes of finite size. Therefore, as a natural extension of the topological derivative concept, we can consider higher order terms in the expansion. In particular, we define the next one as the second order topological derivative. This term provides a more accurate estimation for the size of the holes

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and also it may be used to improve the optimality conditions given by the first order topological derivative (see, for instance, Céa et al., 2000). These features are essential in the context of topology optimization and inverse problems, for instance.

In order to present the basic idea, let us consider an open bounded domain $\Omega \subset \mathbb{R}^2$, with a smooth boundary $\partial\Omega$ and a cost function $\psi(\Omega)$. If the domain Ω is perturbed by introducing a small hole B_ε of radius ε at an arbitrary point $\hat{\mathbf{x}} \in \Omega$, we have a new domain $\Omega_\varepsilon = \Omega \setminus \overline{B}_\varepsilon$, whose boundary is denoted by $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$. From these elements, the topological asymptotic expansion of the cost function may be expressed as

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f_1(\varepsilon)D_\mathrm{T}\psi + f_2(\varepsilon)D^2_\mathrm{T}\psi + \mathscr{R}(f_2(\varepsilon)), \tag{1}$$

where $f_1(\varepsilon)$ and $f_2(\varepsilon)$ are positive functions that decreases monotonically such that $f_1(\varepsilon) \to 0$, $f_2(\varepsilon) \to 0$ when $\varepsilon \to 0^+$ and

$$\lim_{\varepsilon\to 0}\frac{f_2(\varepsilon)}{f_1(\varepsilon)} = 0, \quad \lim_{\varepsilon\to 0}\frac{\mathscr{R}(f_2(\varepsilon))}{f_2(\varepsilon)} = 0. \tag{2}$$

Dividing Eq. (1) by $f_1(\varepsilon)$ and after taking the limit $\varepsilon \to 0$ we obtain

$$D_\mathrm{T}\psi = \lim_{\varepsilon\to 0}\frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f_1(\varepsilon)}, \tag{3}$$

where term $D_\mathrm{T}\psi$ is classically defined as the (first order) topological derivative of ψ. In addition, if we divide Eq. (1) by $f_2(\varepsilon)$ and after taking the limit $\varepsilon \to 0$, we can recognize term $D^2_\mathrm{T}\psi$ as the second order topological derivative of ψ, which is given by

$$D^2_\mathrm{T}\psi = \lim_{\varepsilon\to 0}\frac{\psi(\Omega_\varepsilon) - \psi(\Omega) - f_1(\varepsilon)D_\mathrm{T}\psi}{f_2(\varepsilon)}. \tag{4}$$

In this work we apply the topological-shape sensitivity method developed in Novotny et al. (2003) as a systematic approach to calculate first as well as second order topological derivative for the Poisson's equations, taking the total potential energy as cost function and the state equation as constraint. Furthermore, we also study the effects of different boundary conditions on the hole: Neumann and Dirichlet (both homogeneous). Finally, we present some numerical experiments showing the influence of the second order topological derivative in the topological asymptotic expansion, which has two main features: it allows us to deal with hole of finite size and provides a better descent direction in optimization process.

2. Topological-shape sensitivity method

In Novotny et al. (2003) was proposed an alternative procedure to calculate the (first order) topological derivative called topological-shape sensitivity method. This approach makes use of the whole mathematical framework (and results) developed for shape sensitivity analysis (see, for instance, the pioneering work of Murat & J. Simon (1976)). The main result obtained in Novotny et al. (2003) is given by the following theorem:

Theorem 1. *Let $f_1(\varepsilon)$ be a function chosen in order to $0 < |D_T\psi| < \infty$, then the (first order) topological derivative given by Eq. (3) can be written as*

$$D_\mathrm{T}\psi = \lim_{\varepsilon\to 0}\frac{1}{f_1'(\varepsilon)}\frac{d}{d\varepsilon}\psi(\Omega_\varepsilon), \tag{5}$$

where the derivative of the cost function with respect to the parameter ε may be seen as its classical shape sensitivity analysis.

A remarkable fact concerning the topological-shape sensitivity method is that it can be easily extended to calculate higher order topological derivatives. In particular, following the same idea presented in Theorem 1, it is straightforward to show that: