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Fractal finite element method based shape sensitivity analysis of mixed-mode fracture

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ABSTRACT

In this paper, a new fractal finite element based method for continuum-based shape sensitivity analysis for a crack in a homogeneous, isotropic, and two-dimensional linear-elastic body subject to mixed-mode (modes I and II) loading conditions, is presented. The method is based on the material derivative concept of continuum mechanics, and direct differentiation. Parametric study is carried out to examine the effects of the similarity ratio, the number of transformation terms, and the integration order on the quality of the numerical solutions. Three numerical examples which include both mode-I and mixed-mode problems, are presented to calculate the first-order derivative of the *J*-integral or stress-intensity factors. The results show that first-order sensitivities of *J*-integral or stress-intensity factors obtained using the proposed method are in excellent agreement with the reference solutions obtained using the finite-difference method for the structural and crack geometries considered in this study.

1. Introduction

Recently, methods based on fractal geometry concepts to generate infinite number of finite elements around the crack tip to capture the crack tip singularity have been developed or investigated to solve linear-elastic fracture mechanics (LEFM) problems [1–5]. Fractal finite element method (FFEM) is one such method developed for calculating the stress-intensity factors (SIFs) in linear-elastic crack problems. In its original form, the fractal two-level finite element method (FEM) was first proposed by Leung and Su in 1993 [6], which has been successfully applied since its origin, to solve many kinds of crack problems under mode-I and mixed-mode loading conditions [7–14].

Compared with other numerical methods like finite element method (FEM), FFEM has several advantages. First, by using the concept of fractal geometry, infinite finite elements are generated virtually around the crack tip, and hence the effort for data preparation can be minimized. Second, based on the eigenfunction expansion of the displacement fields [15,16], the infinite finite elements that generate virtually by fractal geometry around the crack tip are transformed in an expeditious manner. This results in reducing the computational time and the memory requirement for fracture analysis of cracked structures. Third, no special finite elements and post-processing are needed to determine the SIFs. Finally, as the analytical solution is embodied in the transformation, the accuracy of the predicted SIFs is high.

In addition to the SIFs, the derivatives of the SIFs are often required to predict the probability of fracture initiation and/or instability in cracked structures. Hence, sensitivity analysis of a crack-driving force plays an important role in many fracture-mechanics applications involving the stability and arrest of crack propagation, reliability analysis, parameter identification, or other considerations. For example, the first- and second-order reliability methods [17], frequently used in probabilistic fracture mechanics [18–24], require the gradient and Hessian of the performance function with respect to random parameters. In LEFM, the performance function is built on the SIFs. Hence, both first-and/or second-order derivatives of *J*-integral or SIFs are needed for probabilistic analysis. The evaluation of response derivatives with respect to crack size is a challenging task, since it requires shape sensitivity analysis. Using a brute-force type finite-difference method to calculate the shape sensitivities is often computationally expensive, in that numerous repetitions of deterministic FEM or FFEM analysis may be required for a complete reliability analysis. Furthermore, if the finite-difference perturbations are too large relative to finite element meshes, the approximations can be inaccurate, whereas if the perturbations are too small, numerical truncation errors may become significant. Therefore, an important requirement of some fracture-mechanics applications is to evaluate the rates of SIFs accurately and efficiently.

Consequently, analytical methods based on virtual crack extension [25–30] and continuum shape sensitivity theory [31–36] have emerged. In 1988, Lin and Abel [25] introduced a virtual crack extension technique to calculate the first-order derivative of

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mode-I SIF for a structure containing a single crack. This method maintains all of the advantages of similar virtual crack extension techniques introduced by deLorenzi [26,27], Haber and Koh [28], and Barbero and Reddy [29], but adds a capability to calculate the derivatives of the SIFs. Subsequently, Hwang et al. [30–33] generalized this method to calculate both first- and second-order derivatives for structures with multiple crack systems, axisymmetric stress states, and crack-face and thermal loading. However, this method requires mesh perturbation—a fundamental requirement of all virtual crack extension techniques. For second-order derivatives, the number of elements affected by mesh perturbation surrounding the crack tip has a significant effect on solution accuracy [30–33]. Feijóo et al. [34] applied the concepts of continuum shape sensitivity theory [35] to calculate the first-order derivative of the potential energy. Since the energy release rate (ERR) is the first-order derivative of potential energy, the ERR or SIFs can be calculated using this approach, without any mesh perturbation. Later, Taroco [36] extended this approach to formulate the second-order sensitivity of potential energy to predict the first-order derivative of the ERR. However, this presents a formidable task, since it involves calculation of second-order stress and strain sensitivities. To overcome this difficulty, Chen et al. [37,38] invoked the domain integral representation of the J-integral and used the material derivative concept of continuum mechanics to obtain first-order sensitivity of the J-integral for linear-elastic cracked structures. Since this method requires only the first-order sensitivity of a displacement field, it is simpler and more efficient than existing methods. Subsequently, Chen et al. [39] extended their continuum shape sensitivity method for mixed-mode loading conditions. Rao and Rahman [40,41] developed a sensitivity analysis method for a crack in an isotropic, linear-elastic functionally graded material under mode-I and mixed-mode loading conditions. However, all of the above methods have been developed in conjunction with FEM.

This paper presents a new FFEM based method for predicting the first-order sensitivity of J-integral or mode-I and mode-II SIFs, K_{I} and K_{II}, respectively, for a crack in a homogeneous, isotropic, and two-dimensional linear-elastic structure subject to mixed-mode (modes I and II) loading conditions. The method is based on the material derivative concept of continuum mechanics, and direct differentiation. Numerical examples are presented to calculate the first-order derivative of the J-integral or SIFs, using the proposed method. The predicted numerical results from this method are compared with those obtained using the finite-difference methods.

2. Fractal finite element method

In FFEM, the domain of a two-dimensional body containing crack is divided into a singular and a regular region, with the regular region enclosing the crack tip and the boundary curve Γ_0 separating the two regions, as shown in Fig. 1. Both the regular and singular regions are modeled using conventional finite elements. With the crack tip as the center of similarity and using ξ as the similarity ratio, an infinite set of curves $\{\Gamma_1, \Gamma_2, \ldots\}$, similar to Γ_0 but with proportional constants $(\xi^1, \xi^2, \ldots)$, are generated inside the singular region. Between the two curves Γ_{k-1} and Γ_k, the region is named the k-th layer. Straight lines that connect the crack tip to the corner nodes lying on Γ_0 are then created, dividing each layer into a mesh of elements with a similar pattern in the process. A fractal mesh is thus generated in the singular region with conventional finite elements only being used. All nodes located on Γ_0 are called the master nodes (m), while those inside Γ_0 are called the slave nodes (s).

2.1. William's eigenfunction expansion

Assuming plane crack with traction-free faces subjected to arbitrary far field loading, the linear elastic displacement field at the crack tip obtained by the William's eigenfunction expansion technique [15] can be expressed as

$$u = \sum_{n=0}^{\infty} \frac{r^{n/2}}{2\mu} \left[\begin{array}{l} a_n^{\mathrm{I}} \left(\left(\kappa + \frac{n}{2} + (-1)^n \right) \cos \frac{n}{2}\theta - \frac{n}{2} \cos \left(\frac{n}{2} - 2 \right) \theta \right) \\ -a_n^{\mathrm{II}} \left(\left(\kappa + \frac{n}{2} - (-1)^n \right) \sin \frac{n}{2}\theta + \frac{n}{2} \sin \left(\frac{n}{2} - 2 \right) \theta \right) \end{array} \right], \tag{1}$$

$$v = \sum_{n=0}^{\infty} \frac{r^{n/2}}{2\mu} \left[\begin{array}{l} a_n^{\mathrm{I}} \left(\left(\kappa - \frac{n}{2} - (-1)^n \right) \sin \frac{n}{2}\theta + \frac{n}{2} \sin \left(\frac{n}{2} - 2 \right) \theta \right) \\ +a_n^{\mathrm{II}} \left(\left(\kappa - \frac{n}{2} + (-1)^n \right) \cos \frac{n}{2}\theta + \frac{n}{2} \cos \left(\frac{n}{2} - 2 \right) \theta \right) \end{array} \right], \tag{2}$$

where μ is the shear modulus, (r, θ) are the polar coordinates, $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress and $\kappa = 3 - 4\nu$ for plane strain with ν being the Poisson's ratio.

The coefficients $a_n^{\mathrm{I,II}}$ can be determined after imposing loading and other boundary conditions. It should be noted that the first-degree coefficients ($a_1^{\mathrm{I,II}}$) in the series are directly associated with the $r^{-1/2}$ term in the stresses which accounts for the singular stress behavior at the crack tip, whereas the first terms of the displacement series $a_0^{\mathrm{I,II}}$ are associated with rigid body motions. Therefore, the relationship

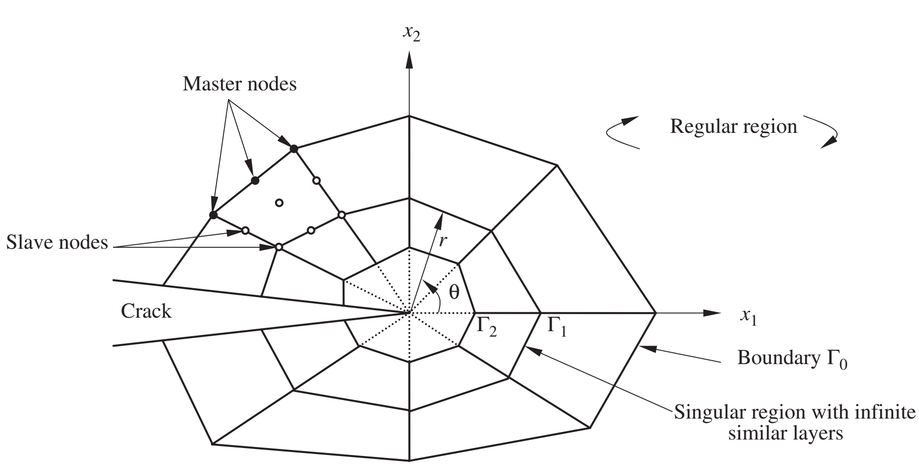

Fig. 1. Cracked body domain with regular region, singular region, and fractal mesh.