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Universal properties of kernel functions for probabilistic sensitivity analysis

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Abstract

Development of probabilistic sensitivities is frequently considered an essential component of a probabilistic analysis and often critical towards understanding the physical mechanisms underlying failure and modifying the design to mitigate and manage risk. One useful sensitivity is the partial derivative of the probability-of-failure and/or the system response with respect to the parameters of the independent input random variables. Calculation of these partial derivatives has been established in terms of an expected value operation (sometimes called the score function or likelihood ratio method). The partial derivatives can be computed with typically insignificant additional computational cost given the failure samples and kernel functions — which are the partial derivatives of the log of the probability density function (PDF) with respect to the parameters of the distribution. The formulation is general such that any sampling method can be used for the computation such as Monte Carlo, importance sampling, Latin hypercube, etc. In this paper, useful universal properties of the kernel functions that must be satisfied for all two parameter independent distributions are derived. These properties are then used to develop distribution-free analytical expressions of the partial derivatives of the response moments (mean and standard deviation) with respect to the PDF parameters for linear and quadratic response functions. These universal properties can be used to facilitate development and verification of the required kernel functions and to develop an improved understanding of the model for design considerations.

Keywords: Probabilistic sensitivities; Mean sensitivity; Variance sensitivity; Score function method; Likelihood ratio method

1. Introduction

Development of probabilistic sensitivities is frequently considered an essential component of a probabilistic analysis and often critical towards understanding the physical mechanisms underlying failure and modifying the design to mitigate and manage risk. Significant progress has been made over the past few decades in developing methods such that the sensitivity information is provided as a by-product of the analysis or at a significantly reduced cost with improved accuracy relative to a finite difference approach. The definition of a probabilistic sensitivity differs, however, between application fields and users, as described below.

Probabilistically-based sensitivity methods have a long and storied history with respect to first and second order reliability methods. Sensitivity factors (derivatives of the safety index with respect to the random variables) [1], derivatives of the probability-of-failure with respect to the random variable parameters [1], and omission factors (relative error in the reliability index when a basic variable is replaced by a deterministic number) [2] are computed as by-products of an analysis.

Variance-based methods [3,4] have been applied to design under uncertainty problems [5,6]. Variance-based methods are capable of identifying the contributions of the "main" and "interaction" effects and the total sensitivity index (the sum of all the sensitivity indices, including all the interaction effects) for any random variable. Groups of variables can also be considered. Calculation of the sensitivity indices requires computation in addition to the probability-of-failure calculation, by requiring multiple multidimensional integrals. The Fourier Amplitude Sensitivity Test (FAST) method can be used to reduce the multidimensional integrals into a one-dimensional integral [7].

Lui and Chen [8] use the Kullback–Leibler relative entropy-based method to evaluate the impact of a random variable on a design performance by measuring the divergence between

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Nomenclature

a	general distribution parameter
b	general distribution parameter
$E[\cdot]$	expected value operator over entire sample space
$E[\cdot]_{\Omega}$	expected value operator over failure domain
$E[\cdot]_{\bar{\Omega}}$	expected value operator over safe domain
f	PDF
F	CDF
I	indicator function (one in the failure region, zero otherwise)
J	Jacobian of transformation between μ, σ and a, b
M_{θ}	kernel modifier for truncated distributions
n	number of random variables
N	number of Monte Carlo samples
V	variance
X	random variable
$\mathbf{X}$	vector of random variables
U	standard normal variate
κ_{θ}	kernel function with respect to an arbitrary distributional parameter θ
$\hat{\kappa}_{\theta}$	kernel function for a truncated distribution
κ_a	kernel function with respect to distributional parameter a
κ_b	kernel function with respect to distributional parameter b
κ_{μ}	kernel function with respect to distributional parameter μ
κ_{σ}	kernel function with respect to distributional parameter σ
z	response function
ψ	digamma function
Γ	gamma function
ϕ	standard normal PDF
Φ	standard normal CDF
μ	mean
Ω	failure domain
$\bar{\Omega}$	safe domain
σ	standard deviation
θ	arbitrary PDF parameter
$\Psi_{\theta}^{(n)}$	integral of moments of normal kernel functions

two probability density functions of the performance response, obtained before and after the variance removal of a random variable. The application is similar to the variance-based methods but is not limited to differences in the second moment.

Helton [9] discusses the use of scatter plots, regression analysis, and Spearman or Pearson correlation and other methods as qualitative and quantitative metrics for sampling methods as inexpensive approaches to discerning the contribution of the variance of each random variable to the output.

Kleijnen and Rubinstein [10], Rubinstein [11], and Rubinstein and Shapiro [12] discuss the "score function" method for the computation of partial derivatives of a performance function with respect to parameters of the underlying probability distributions. The method is applied to discrete event simulations. Glynn developed the equivalent "likelihood ratio" method and also applied it to discrete events [13].

Karamchandani independently presented the fundamental concept of the score and likelihood ratio methods and applied the method to structural reliability problems [14]. He also suggested a non-dimensionalizing approach in order to compare probabilistic sensitivities across random variables.

Wu [15] applied the score function concept to system reliability problems and used importance sampling for the probability calculations.

Wu and Mohanty [16] proposed using the score function methodology as a screening method for problems with a large number of random variables. In their approach, all random variables are mapped to standard normal before obtaining the sensitivities. The sensitivities with respect to the mean and standard deviation then follow t and chi-squared distributions, respectively. Hypothesis testing is used to identify significant variables. Included in this work is the calculation of the sensitivity of the mean of the response to the input PDF parameters.

Sues and Cesare [17] apply the score function approach to system reliability problems and develop sensitivities of the mean and standard deviation of the system response to the input PDF parameters.

There are several factors that make the calculation of partial derivatives appealing using the score function method; primary among them, minimal (usually insignificant) additional computational time is required because the calculation uses pre-existing samples. In addition, the methodology can be used with Monte Carlo and other sampling techniques such as importance sampling or Latin hypercube, does not require any "smoothness" conditions on the limit state, multiple limit states can be considered, and the method can be added with small effort to existing software once the kernel functions are defined.

As pointed out by several authors, the key to applying the score function approach is to develop and apply the kernel functions, i.e., the partial derivatives of the log of the PDF with respect to parameters of the PDF; however, no study has been reported on the properties of these kernel functions. In this research, universal properties of the kernel functions that must be satisfied for all two parameter probability distributions are derived. These properties are then used to develop analytical sensitivity results for linear and quadratic response functions and to develop insight and understanding of the probabilistic response of the model. A numerical example consisting of two random variables (normal and EVD) and a quadratic response function and limit state is presented.

2. Methodology

The probability-of-failure is computed as

$$P_f = \int_{g(\mathbf{x}) \leq 0} f_x(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} \tag{1}$$

where $\mathbf{x}$ is a vector of random variables, $f_{\mathbf{X}}(\mathbf{x})$ is the joint density function of $\mathbf{x}$, and g is the limit state function defined such that $g(\mathbf{x}) \leq 0$ defines failure.