Computers and Mathematics with Applications 58 (2009) 1239–1246

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Sensitivity analysis in convex programming

P. Jiménez Guerra [a,*], M.A. Melguizo [b], M.J. Muñoz-Bouzo [a]

[a] Departamento de Matemáticas Fundamentales, Facultad de Ciencias, Universidad Nacional de Educación a Distancia, Senda del Rey 9, 28040 Madrid, Spain

[b] Departamento de Matemática Aplicada, Universidad de Alicante, Carretera San Vicente del Raspeig, s/n, 03690 San Vicente del Raspeig, Alicante, Spain

ARTICLE INFO

Article history:
Received 16 May 2008
Received in revised form 2 June 2009
Accepted 23 June 2009

Keywords:
Continuity
Derivative
Set-valued map
Vector optimization

ABSTRACT

The object of this paper is to perform an analysis of the sensitivity for convex vector programs with inequality constraints by examining the quantitative behavior of a certain set of optima according to changes of right-hand side parameters included in the program. The results in the paper prove that the sensitivity of the program depends on the solution of a dual program and its sensitivity.

1. Introduction

The problem of analyzing the sensitivity in vector programming has drawn the attention of many authors from the works by Kuhn and Tucker in 1951. It is known that one of the difficulties that appears treating this problem lies in the fact that, in vector programming, the set of the optimal values most of times is not a singleton. Thus whereas in the scalar case of scalar programming the optimal value reached is a minimum value, and therefore unique, in the case of vector programming the optimal ones are multiple values. This implies that, in general, it turns out to be more complicated to analyze sensitivity in vector optimization programs than in scalar optimization programs, since in the vectorial case the analysis of the sensitivity may necessitate to study a set-valued map (the set-valued map that assigns to each value of some parameter the set of optimal values reached by its associated program) whilst in the scalar case the analysis of sensitivity with respect to a parameter consists of the study of a function (the function that assigns to each value of the parameter the minimum value reached by its associated program).

One of the techniques used in sensitivity analysis is to reduce the problem by choosing a particular point in the efficient set. This is the case if we are interested in the best alternative which minimizes a specific scalar utility function as in [1], where the authors reduce to an optimization problem with scalar objective by minimizing the distance between some fixed desirable point and the efficient set, or as in [2], where the scalarization is done by the weighted sum approach, etc.

When dealing with a subset or the whole set of efficient points, there are several procedures. One is to assume the existence of an adequate selection of particular efficient points as in [3] where the authors study sensitivity taking a selection of the balance points introduced by E. Galperin and further developed in [4]. In [5–9] the authors consider the so-called T-optimal solutions and also assume the existence of a Fréchet differentiable selection. However, there are several approaches which deal with sets of efficient points and focus on the behavior of some set-valued perturbation maps (e.g., [10–12], the two survey papers [13,14], and the references therein).

* Corresponding author.
E-mail addresses: pjimenez@mat.uned.es (P.J. Guerra), miguelangel.melguizopadial@telefonica.es (M.A. Melguizo), mjmunoz@mat.uned.es (M.J. Muñoz-Bouzo).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.06.041

Continuing the line of inquiry of [7], a sensitivity analysis is performed in [15] for differential vector programs with equality constraints with respect to the right-hand side, proving that sensitivity of the problem depends not only on a suitable Lagrange multiplier but also on the derivative of a set-valued function of Lagrange multipliers.

Following a similar procedure to [15], the present paper performs an analysis of sensitivity for convex vector programs with inequality constraints, analyzing the quantitative behavior of certain set of optimal points (that are dense in the efficient line) according to changes of the right-hand side parameters included in the program.

The paper is organized as follows. Section 2 introduces notation, basic concepts, and some results that will be used throughout the paper. Section 3 is devoted to study two important properties (Theorems 4 and 5) of the set-valued function being the solution of a dual program that constitutes the base for the analysis of sensitivity developed in the next section. In Section 4, Theorem 6 proves that the sensitivity of the program depends on the set-valued function studied in Section 3 and the sensitivity of this set-valued function. A particularization of Theorem 6 for p-homogeneous programs is also presented in Theorem 7, because of its special usefulness since linear programs are 1-homogeneous.

2. Notations and preliminaries

Throughout the paper let X, Y, Z and W be Banach spaces such that Y and Z are respectively ordered by closed convex pointed cones Y_+ and Z_+ where Y_+ is also pointed, and W is an order complete, i.e., every non-empty bounded from below subset has an infimum in W, Banach lattice with positive cone W_+. Let us consider a linear and surjective continuous mapping $T : Y \to W$ such that T is strictly positive, i.e., $T(Y_+ \setminus \{0\}) \subset W_+ \setminus \{0\}$, and $\operatorname{Ker} T$ has a topological supplement Y_T in Y. In particular, if Y is a Hilbert space then the orthogonal complement of $\operatorname{Ker} T$ is a topological supplement of $\operatorname{Ker} T$ in Y. Let $\widehat{T}$ denote the restriction of T to Y_T. It follows from the open mapping theorem [16, Theorem 2.11] that the inverse operator $\widehat{T}^{-1}$ is continuous. Let us denote by $\mathfrak{L}(Z, W)$ the space of all linear and continuous mappings from Z into W endowed with the usual norm and order.

Finally, let be $B \subset X$ a convex subset, $V \subset Z$ an open convex set such that $0 \notin V$, $b \in V$, two convex functions $f : B \to Y$ and $g : B \to Z$ and the following optimization problem:

$$\left.\begin{array}{c}\operatorname{Min} f(x)\\ x \in D, g(x) \le b\end{array}\right\} \quad (1_b).$$

Following [5], $x_b \in X$ is said to be a T-optimal solution of (1_b) if x_b is feasible and $Tf(x_b) \le Tf(x)$ for every feasible $x \in B$. A map $L \in \mathfrak{L}(Z, W)$ is said to be a Lagrange T-multiplier of (1_b) if $L \ge 0$, i.e., $L(Z_+) \subset W_+$, and

$$\inf \{Tf(x) \mid x \in B, g(x) \le b\} = \inf \{Tf(x) + L(g(x) - b) \mid x \in B\}.$$

Theorem 3 in [5] states that if there exists a T-optimal solution of (1_b) and the following Slater condition is fulfilled, there exists $x_1 \in B$ such that $g(x_1) \in -\operatorname{int}(Z_+)$, then there exists a Lagrange T-multiplier of (1_b). Let us denote by Γ_T the set of all $L \in \mathfrak{L}(Z, W)$ such that $L \ge 0$ and the set

$$\{Tf(x) + Lg(x) \mid x \in B\}$$

is bounded from below in W. Set

$$\varphi(T, L) = \operatorname{Inf} \{Tf(x) + Lg(x) \mid x \in B\}$$

for every $L \in \Gamma_T$ and the dual function

$$\psi(T, G) = \widehat{T}^{-1} \varphi(T, TG)$$

for every $G \in \mathfrak{L}(Z, Y)$ such that $TG \in \Gamma_T$.

Now we can define the following dual program:

$$\left.\begin{array}{c}\operatorname{Max} \psi(T, G) - G(b)\\ G \in \mathfrak{L}(Z, Y), TG \ge 0\\ TG \in \Gamma_T\end{array}\right\} \quad (2_b)$$

$G_b \in \mathfrak{L}(Z, Y)$ is said to be a T-optimal solution of (2_b) if G_b es feasible and

$$T(\psi(T, G) - G(b)) \le T(\psi(T, G_b) - G_b(b))$$

for every feasible $G \in \mathfrak{L}(Z, Y)$. If x_b is a T-optimal solution of (1_b), G_{x_b} is a T-optimal dual solution of (2_b) and

$$f(x_b) = \psi\left(T, G_{x_b}\right) - G_{x_b}(b)$$

we then say that $\left[x_b, G_{x_b}\right]$ are T-optimal associated solutions. Theorem 5 in [5] proves that if x_b is a T-optimal solution of (1_b) then there exists a T-optimal dual solution $G_{x_b} \in \mathfrak{L}(Z, Y)$ of (2_b) such that $\left[x_b, G_{x_b}\right]$ are associated T-optimal solutions if and only if there exists a Lagrange T-multiplier $L_b \in \mathfrak{L}(Z, W)$ of (1_b). Then the equality $L_b = TG_{x_b}$ holds.

We recall some basic definitions dealing with differentiable set-valued maps that we will use throughout the paper. For further information about set-valued analysis see, for instance, the book of [17].

Let S be a normed space, $A \subset S$ a non empty set and $\overline{A}$ its closure in the norm topology. Let $x \in \overline{A}$. The contingent cone $T_A(x)$ and the adjacent cone $T_A^{\flat}(x)$ are defined by