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Sensitivity analysis in kinematically incompatible models

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ABSTRACT

A natural question that arises when working with the so-called kinematically incompatible models is about the positioning of the internal artificial boundary that splits the domain of analysis into the sub-domains for which the different, incompatible, models are established. Although the experience in modeling might provide an *ad hoc* solution to such matter it may also draw misleading conclusions. Here, the sensitivity analysis furnishes a systematic and reliable framework to study the impact of the positioning of the artificial boundary over the solution of the problem. In this sense, the present work is concerned with the sensitivity analysis of the variational formulation corresponding to the kinematically incompatible models when the boundary over which the incompatibility between the models takes place is changed. Assuming the presence of a discontinuity in the fields over a given interface between two incompatible models, this analysis allows to measure the sensitivity of a given cost functional when the location of such interface is changed. The application for which this analysis is envisaged is to assess the correctness, or incorrectness, in the definition of the positioning of such coupling interface between dimensionally-heterogeneous domains. Also, some numerical results are provided to give numerical evidence of the usefulness of the present tool.

1. Introduction

From its birth 30 years ago, the sensitivity analysis has proven to be a valuable tool to evaluate how sensible a cost functional associated to a given problem is in front of perturbations in the definition of such problem [1–3] (and references therein). This kind of analysis can be encountered under different names according to the nature of the perturbation. Thus, when the parameters that define the problem are perturbed it is referred to as parameter sensitivity analysis [2]. If the perturbations are performed along the boundary of the domain of analysis then it is referred to as shape sensitivity analysis [1,4,5]. Finally, when the perturbations are singular such as modifications in the topology of the domain of analysis or discontinuous perturbations in some of the parameters that define the problem, then it is referred to as topological sensitivity analysis [6–8]. It is worthwhile to recall that in any of the contexts mentioned above, the variational calculus provides a consistent framework to carry out the calculation of the sensitivity, revealing in a straightforward manner a series of underlying concepts.

Kinematically incompatible models are a class of mathematical models that allow the fields in the problem to be discontinuous over a given internal boundary. The role of this internal boundary, actually artificial from the physical standpoint, is to establish a division in the nature of the model when thinking in terms of its kinematics. In the simplest case, a kinematically incompatible model consists of a partition of the domain of analysis into two sub-regions. Over each sub-domain a given kinematics is defined, giving rise to two different sub-models that share a common internal boundary but that are ruled by different kinematics. A theoretical account including the extended variational principles for such models was introduced in [9,10] for fluid and solid mechanics, respectively.

A natural question that arises when working with incompatible models, when looking at the simple example involving two sub-domains, is about the positioning of the artificial boundary that splits the domain of analysis into the two sub-domains for which the different, incompatible, models will be established. Clearly, a wrong placement of such artificial boundary would produce incorrect solutions because the problem is not appropriately represented by the kinematical incompatible model. Since such internal boundary is artificial, it is not desirable that its position affects significantly the solution of the problem. Although the experience in modeling might provide an *ad hoc* solution to such matter it may also draw misleading conclusions. Hence, the need for a systematic and reliable framework is compulsory, and the sensitivity analysis furnishes such a framework in order to study the impact that a change like the one mentioned before produces over the solution of the problem. Recall that the incompatibility in the

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kinematics entails a discontinuity in the fields [9,10]. Therefore, the sensitivity analysis to the change in the position of the internal boundary can be understood in two ways: (i) as the sensitivity to the shape change of both domains taking care that they move in an attached fashion, that is at the same time, or (ii) as the sensitivity to the displacement of the discontinuity that is consequence of the incompatibility. Thus, in this work we make use of the concept of sensitivity analysis in a problem involving kinematically incompatible models so as to assess the correctness, or not, in the partitioning of a domain of analysis into two sub-domains for which different models will be used.

This work is organized as follows: in order to make the paper self-contained we present in Section 2 the bases for the extended variational principle for kinematically incompatible models through its application to the heat transfer problem. Readers interested in more details are invited to see [9,11]. Section 3 takes this variational formulation and carries through the sensitivity analysis to the change in the positioning of the discontinuity product of the incompatibility. Section 4 provides some numerical evidences of the usefulness of the analysis performed here. Finally, Section 5 closes the work with some final remarks and conclusions.

2. Kinematically incompatible models

This section gives a brief account of kinematically incompatible models in the particular case of the problem of coupling a 3D model with a 1D model for the heat transfer problem. In the final part the role of the sensitivity analysis is commented in order to motivate the developments of the forthcoming sections.

2.1. Extended variational principle

Consider the heat transfer problem in a domain $\Omega \subset \mathbb{R}^3$ as shown in Fig. 1. Such figure also shows a partition of Ω given by an internal boundary Γ_a, that is $\Omega = (\overline{\Omega_1 \cup \Omega_2})^\circ$.

The classical variational principle poses the problem of finding a scalar field $\theta \in \mathscr{U}$ such that it satisfies a variational equation, where $\mathscr{U}$ is the space $H^1(\Omega)$ plus essential boundary conditions. A kinematical incompatibility in a model arises when a kinematical hypothesis is taken over a partition of Ω, while the kinematics for the complementary partition remains invariant (in the simple case of a partition of two sub-domains). In this situation, the field θ is now a pair (θ_1, θ_2), where both fields have their own characteristics. In the problem presented here, due to the particular form of Ω_1, over that sub-domain it is possible to make the assumption of a constant scalar field over each transversal section of the domain. Naming z the axial coordinate, the field θ_1 is just a function of that coordinate and the problem can be reduced to a 1D problem over Ω_1 that now ranges in the interval (z_a, z_b) (along the z axis) as shown in Fig. 2. Over Ω_2 the field θ_2 is considered to have a three dimensional description.

The extended governing variational principle for this problem, in the steady state case, reads as follows (see [11]): for some $\gamma \in [0,1]$ find $((\theta_1,\theta_2),t_1,t_2) \in \mathscr{U}_1 \times \mathscr{U}_2 \times \mathbb{R} \times H^{-1/2}(\Gamma_a)$ such that

$$\begin{aligned}
&\int_{z_a}^{z_b}\left[Ak\frac{\mathrm{d}\theta_1}{\mathrm{d}z}\frac{\mathrm{d}\eta_1}{\mathrm{d}z} - Af\eta_1\right]\mathrm{d}z + \int_{\Omega_2}[k\nabla\theta_2\cdot\nabla\eta_2 - f\eta_2]\,\mathrm{d}\mathbf{x}\\
&\quad+\gamma t_1\left(\eta_1 - \frac{1}{A_a}\int_{\Gamma_a}\eta_2\,\mathrm{d}\Gamma_a\right)\Big|_{z_b} + (1-\gamma)\int_{\Gamma_a}t_2(\eta_1-\eta_2)\,\mathrm{d}\Gamma_a\\
&\quad+\gamma s_1\left(\theta_1 - \frac{1}{A_a}\int_{\Gamma_a}\theta_2\,\mathrm{d}\Gamma_a\right)\Big|_{z_b} + (1-\gamma)\int_{\Gamma_a}s_2(\theta_1-\theta_2)\,\mathrm{d}\Gamma_a\\
&\quad= 0 \quad \forall((\eta_1,\eta_2),s_1,s_2) \in \mathscr{V}_1 \times \mathscr{V}_2 \times \mathbb{R} \times H^{-1/2}(\Gamma_a),
\end{aligned} \tag{1}$$

where $A = A(z)$ is the cross-sectional area in Ω_1, k is the conductivity, f is a volume source, A_a is the measure of boundary Γ_a, and was assumed that the material has an isotropic constitutive behavior such that follows the Fourier law in both domains. Over Ω_1 both k and f have been assumed constants across the transversal area. Also it is

$$\begin{aligned}
&\mathscr{U}_1 = \{\theta_1 \in H^1(]z_a,z_b[);\, \theta_{1|z_a} = \bar{\theta}_1\},\\
&\mathscr{U}_2 = \{\theta_2 \in H^1(\Omega_2);\, \theta_{2|\Gamma_{D2}} = \bar{\theta}_2\},
\end{aligned} \tag{2}$$

while $\mathscr{V}_1$ and $\mathscr{V}_2$ are the spaces obtained from differences between elements in $\mathscr{U}_1$ and $\mathscr{U}_2$, respectively.

The set of Euler–Lagrange equations associated to the variational problem (1) are the following:

$$\begin{cases}
-\frac{\mathrm{d}}{\mathrm{d}z}\left(Ak\frac{\mathrm{d}\theta_1}{\mathrm{d}z}\right) = Af & \text{in } (z_a,z_b),\\
-\mathrm{div}\,(k\nabla\theta_2) = f & \text{in } \Omega_2,\\
\theta_1 = \bar{\theta}_1 & \text{at } \{z_a\},\\
\theta_2 = \bar{\theta}_2 & \text{over } \Gamma_{D2},\\
k\nabla\theta_2\cdot\mathbf{n}_2 = 0 & \text{over } \Gamma_{N2},\\
\gamma\left(\theta_1 - \frac{1}{A_a}\int_{\Gamma_a}\theta_2\,\mathrm{d}\Gamma_a\right) = 0 & \text{at } \{z_b\},\\
(1-\gamma)(\theta_1-\theta_2) = 0 & \text{over } \Gamma_a,\\
\gamma t_1 + (1-\gamma)\int_{\Gamma_a}t_2\,\mathrm{d}\Gamma_a = -A_a k\frac{\mathrm{d}\theta_1}{\mathrm{d}z} & \text{at } \{z_b\},\\
\gamma t_1 + (1-\gamma)A_a t_2 = A_a k\nabla\theta_2\cdot\mathbf{n}_2 & \text{over } \Gamma_a.
\end{cases} \tag{3}$$

Remark 1. The role of the real parameter γ is to provide the sense in which the continuity of the quantities is satisfied. Thus, $\gamma = 1$ establishes the continuity of the heat flux over Γ_a in a pointwise sense (strong sense), whereas the continuity of θ is ensured in a mean sense (weak sense). The opposite happens for $\gamma \in [0,1)$ since the continuity of the heat flux is given in a weak sense, while the field θ is pointwisely continuous over Γ_a. These comments must be understood in the sense given by the Euler–Lagrange equations (3). Therefore, the choice of γ is a decision that should be related to the physical characteristics of the problem that is being dealt with.

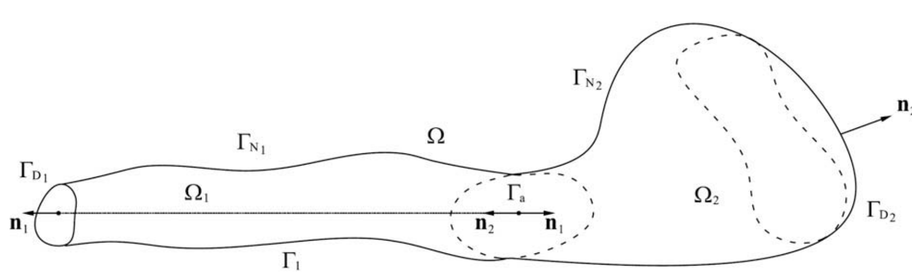

Fig. 1. Domain of analysis Ω and decomposition into Ω_1 and Ω_2.