Computers and Chemical Engineering xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

Computers and Chemical Engineering

journal homepage: www.elsevier.com/locate/compchemeng

Constrained dynamic programming of mixed-integer linear problems by multi-parametric programming

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A R T I C L E I N F O

Article history:
Received 3 September 2013
Received in revised form 9 March 2014
Accepted 26 March 2014
Available online xxx

Keywords:
Dynamic programming
Multi-parametric programming
Mixed-integer linear programming

A B S T R A C T

This work addresses the topic of constrained dynamic programming for problems involving multi-stage mixed-integer linear formulations with a linear objective function. It is shown that such problems may be decomposed into a series of multi-parametric mixed-integer linear problems, of lower dimensionality, that are sequentially solved to obtain the globally optimal solution of the original problem. At each stage, the dynamic programming recursion is reformulated as a convex multi-parametric programming problem, therefore avoiding the need for global optimisation that usually arises in hard constrained problems. The proposed methodology is applied to a problem of mixed-integer linear nature that arises in the context of inventory scheduling. The example also highlights how the complexity of the original problem is reduced by using dynamic programming and multi-parametric programming.

1. Introduction

Multi-stage decision processes occur in different fields of study, such as energy planning (Pereira and Pinto, 1991), computational finance (Seydel, 2012), computer science (Leiserson et al., 2001), or optimal control (Bertsekas, 1995).

Dynamic programming (Bellman, 1957; Powell, 2007; Sniedovich, 2010) is an optimisation theory used to efficiently obtain optimal solutions for problems involving multi-stage decision processes. In industrial applications, dynamic programming has been used to address problems related to project scheduling (Choi et al., 2004; Herroelen and Leus, 2005), optimal control (Dadebo and McAuley, 1995; Bertsekas, 1995) or robust control (Nilim and El Ghaoui, 2005; Kouramas et al., 2012).

The method is based on the principle of optimality, and aims to reduce the complexity of conventional optimisation techniques, by exploring the sequential structure inherent to multi-stage optimisation problems. Given this structure, a suitable recursion is formulated, and the original problem disassembles into a set of sub-problems of lower dimensionality that may be solved sequentially. The dynamic programming recursion preserves the information regarding the impact of current decisions in future stages, while reducing the computational effort required to solve the overall problem.

Despite the advantages of using dynamic programming for multi-stage decision processes, its use is limited in the presence of hard constraints. In this case, at each stage of the dynamic programming recursion, non-linear decisions result and global optimisation procedures are required to solve the dynamic programming problem (Faísca et al., 2008). Another important challenge in constrained dynamic programming is that the computation and storage requirements significantly increase in the presence of hard constraints (Bertsekas, 1995).

To address these issues, different algorithms for constrained dynamic programming have been proposed, combining the principle of optimality of Bellman (1957) and multi-parametric programming techniques (Dua and Pistikopoulos, 2000; Bemporad et al., 2002). By combining these techniques with the principle of optimality, the issues that arise for hard constrained problems are handled in a systematic way, and the shortcomings of conventional dynamic programming techniques are avoided. This method has been used to address constrained dynamic programming problems involving linear/quadratic models (Borrelli et al., 2005; Faísca et al., 2008), and mixed-integer linear/quadratic models (Borrelli et al., 2005). However, as noted by Faísca et al. (2008), the latter involves a reformulation that produces non-convex cost functions, and therefore global optimisation techniques are required.

The proposed work extends the approach of Faísca et al. (2008) to constrained dynamic problems involving mixed-integer formulations with linear objective function and linear constraints. The proposed algorithm involves reformulating the dynamic programming recursion as a mixed-integer linear problem. At each iteration

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http://dx.doi.org/10.1016/j.compchemeng.2014.03.021
0098-1354/

of the recursion, the original problem is relaxed by only considering the set of constraints related to the current stage. Furthermore, by taking the discrete and continuous decision variables as the optimisation vector, and the current state and future optimisation variables as parameters, the cost function of the stage is convex and the need for global optimisation techniques is avoided.

The paper is organised as follows. Section 2 presents the background on dynamic programming and shows how the corresponding recursion may be formulated as a multi-parametric mixed-integer problem, when the system is described by linear dynamics and linear constraints. It also describes a solver suitable for multi-parametric mixed integer linear problems, and the properties of the solution obtained by multi-parametric programming. The material in this section serves as the basis for the algorithm for constrained dynamic programming of mixed-integer linear systems presented in Section 3. Section 4 presents a numeric example, in which the proposed algorithm is applied to an inventory scheduling problem. Qualitative considerations regarding the complexity of this algorithm, compared to conventional multi-parametric programming, are also presented in Section 4, while the overall conclusions of the work are summarised in Section 5.

2. Constrained dynamic programming and multi-parametric programming

Dynamic programming is a technique used to efficiently solve problems involving constrained multi-stage decision processes. Such problems may have a variety of different formulations, but, in general, involve a stage-additive formulation such as follows (Bertsekas, 1995).

$$\begin{aligned} z_k(s_{k-1}) = \quad & \min_{x,y} f_N(x_{N-1}, y_{N-1}, s_N) + \sum_{i=k}^{N-1} f_i(x_{i-1}, y_{i-1}, s_i) \\ \text{s.t.} \quad & g_j(x_{j-1}, y_{j-1}, s_{j-1}) \le 0, \quad j = k, \ldots, N \\ & h_j(x_{j-1}, y_{j-1}, s_{j-1}) = 0, \quad j = k, \ldots, N \end{aligned} \tag{1}$$

Problem (1) comprises $N-k$ decision steps, each involving continuous, $x_i \in \mathbb{R}^{n_c}$, and discrete, $y_i \in \{0, 1\}^{n_d}$, decision variables that influence the state $s_i \in \mathbb{R}^{n_s}$. The formulation is subject to a set of inequality constraints, $g_j(x_{j-1}, y_{j-1}, s_{j-1}) \le 0$, equality constraints, $h_j(x_{j-1}, y_{j-1}, s_{j-1}) = 0$, and a stage cost, $f_i(x_{i-1}, y_{i-1}, s_i)$. The cost function of the multi-stage problem (1) is given by $z_k(s_{k-1}) \in \mathbb{R}$.

The sequential structure of (1) may be explored by applying the optimality principle, proposed by Bellman (1957). In general terms, this principle states that, given an optimal path, V_i, from i to N, the optimal path from k to N, that passes by i, also contains the path V_i. When applied to (1), the optimality principle results in the following recursion.

$$\begin{aligned} z_i(s_{i-1}) = \quad & \min_{x_{i-1}, y_{i-1}} f_i(x_{i-1}, y_{i-1}, s_i) + z_{i+1}(s_i) \\ \text{s.t.} \quad & g_i(x_{i-1}, y_{i-1}, s_{i-1}) \le 0 \\ & h_i(x_{i-1}, y_{i-1}, s_{i-1}) = 0 \end{aligned} \tag{2}$$

Note that problem (2) is a sub-problem of (1), with decision variables, and constraints, pertaining only to stage k. Bellman (1957) demonstrated that, by recursively solving (2) for $i = N, \ldots, k$, the globally optimal solution of (1) is obtained.

We address a special case of (1) and (2) where the objective function and constraints are linear, as follows.

$$\begin{aligned} z_i(s_{i-1}) = \quad & \min_{x_{i-1}, y_i} Cx_{i-1} + Dy_{i-1} + z_{i+1}(s_i) \\ \text{s.t.} \quad & Ax_{i-1} + Ey_{i-1} \le b \\ & Lx_{i-1} + Ky_{i-1} = c \\ & y_{i-1} \in \{0, 1\}^{n_d} \end{aligned} \tag{3}$$

The matrices A, C, D, E, L, K, and vectors b, c in (3) are linear coefficients of appropriate dimensions.

Note that although y in (3) is a binary vector, the formulation may be used to describe an integer vector, $y_i \in (I \in \mathbb{Z}^{n_d})$, by introducing n_d binary variables, $d_i \in \{0, 1\}^{n_d}$, such that $y_i = \sum_{j=1}^{n_d} I_j d_{i,j}$, and $\sum_{j=1}^{n_d} d_{i,j} = 1$.

The method for constrained dynamic programming by multi-parametric programming proposed by Faísca et al. (2008), for multi-parametric linear/quadratic problems, involves the following recursion, formulated for stage i of a process with N stages.

$$\begin{aligned} z_i(\theta_i) = \quad & \min_{x_{i-1}} f_i(x_{i-1}, \theta_i) + z_{i+1}(\theta_{i+1}) \\ \text{s.t.} \quad & g_i(x_{i-1}, \theta_i) \le 0 \\ & h_i(x_{i-1}, \theta_i) = 0 \\ & \theta_i = [s_i, x_i, \ldots, x_{N-1}] \end{aligned} \tag{4}$$

By considering a parameter vector, θ_i, that includes both the state vector, s_{i-1}, and the future decisions, $x_i, \ldots, x_{N-1}$, a convex objective function is obtained, and problem (4) may be solved without the need for global optimisation techniques.

Following the same methodology for the special case of multi-parametric mixed-integer linear programming problems, such as (3), a series of problems of the general form (5) is obtained.

$$\begin{aligned} z_i(\theta_i) = \quad & \min_{x_{i-1}, y_{i-1}} Cx_{i-1} + Dy_{i-1} \\ \text{s.t.} \quad & Ax_{i-1} + Ey_{i-1} \le b + F\theta_i \\ & Lx_{i-1} + Ky_{i-1} = c + Q\theta_i \\ & \theta_i \in \Theta_i, \quad y_{i-1} \in \{0, 1\}^{n_d} \end{aligned} \tag{5}$$

In (5), Θ_i corresponds to a feasible set of initial states. Note that the vector of parameters θ_i is augmented to include the discrete optimisation variables, $y_{i+1}, \ldots, y_N$.

To obtain the solution of problem (5), several solving algorithms exist in the literature (Acevedo and Pistikopoulos, 1997; Pertsinidis et al., 1998; Dua and Pistikopoulos, 2000; Li and Ierapetritou, 2007; Wittmann-Hohlbein and Pistikopoulos, 2012).

The algorithm proposed by Dua and Pistikopoulos (2000) is based on the decomposition of problem (5) into a mp-LP subproblem, and an associated deterministic MILP problem. This algorithm is outlined below.

The mp-LP subproblem associated with (5) is obtained by fixing $y_i = \bar{y}_i$, where $\bar{y}_i$ is a feasible solution of (5).

The procedure for solving the mp-LP subproblem involves analysing the optimality conditions in the neighbourhood of an optimal solution, for a perturbation in the parameter vector, as outlined below.

Please cite this article in press as: Rivotti P, Pistikopoulos EN. Constrained dynamic programming of mixed-integer linear problems by multi-parametric programming. Computers and Chemical Engineering (2014), http://dx.doi.org/10.1016/j.compchemeng.2014.03.021