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# A dimension-reduced method of sensitivity analysis for stochastic user equilibrium assignment model

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ABSTRACT

This paper gives a new dimension-reduced method of sensitivity analysis for perturbed stochastic user equilibrium assignment (SUEA) model based on the relation between its Lagrange function and logarithmic barrier function combined with a Courant quadratic penalty term. The advantage of this method is of smaller dimension than general sensitivity analysis and reducing complexity. Firstly, it presents the dimension-reduced sensitivity results of the general nonlinear programming perturbation problem and the improved results when the objective or constraint functions are not twice continuously differentiable. Then it proves the corresponding conclusion of SUEA with smooth or non-smooth cost functions by the method of converting constraint conditions and decision variables. Finally, two corresponding examples (smooth and non-smooth) are given to illustrate the feasibility of this method.



## 1. Introduction

Sensitivity analysis is the problem about the approximation of solution with respect to parameters, including computing the partial derivatives and gradient of solution to the perturbation parameters. Sensitivity analysis for differential equation, difference equation, algebraic equation, linear programming, nonlinear programming and variational inequality is common in papers.

Generally speaking, sensitivity analysis has several effects as follows:

1. solving new perturbed problem;
2. obtaining the first-order approximation solution;
3. evaluating the parameters' sensitivity and selecting proper parameter;
4. applying the results to predict practical problem;
5. reducing to find some new theoretic problems.

Generally, we use sensitivity analysis of nonlinear programming or variational inequation to study some problems, such as economic models, operational research, traffic assignment, and signal control problem. In this paper, we mainly discuss the sensitivity analysis for traffic assignment models, which we use to obtain its following effects: the first, computing the sensitivity of perturbation parameters such as signal splits, ramp metering rates and traffic demands; the second, predicting

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the perturbed problem so as to improving the road network and control strategies; finally, constructing some efficient algorithms of bi-level traffic problems, which could reduce the difficulty of the NP hard problems. So far, many authors have researched on sensitivity analysis for some mathematical systems, the more important of which are [1,2]. In [1] the author gave the sensitivity analysis theorem of second-order differentiable unique local solution and presented the special sensitivity analysis results based on penalty function method. Some theorems about sensitivity analysis for many mathematical systems were presented in [2] and applied in [3]. Additionally, there are a lot of papers giving the sensitivity information for some special mathematical programs, for example, Qiu and Magnanti [4] introduced some sensitivity analysis conclusions for variational inequation defined on polyhedral sets and Dafermos [5] gave sensitivity theorems of equilibrium network models with linear demand-supply function. Sensitivity analysis method has been applied to traffic assignment models and signal control models in many articles [11–17]. One important is Tobin and Friesz [3], where the author applied the theorems in [2] to the restricted problem equivalent to the original traffic assignment model and gave gradient information of the equilibrium path flow with respect to several perturbation parameters. The derivatives of delay function in signal control problem were obtained by some heuristic algorithms in [6]. In [7], the application of sensitivity analysis on bi-level program was mainly discussed and detailed examples corresponding to different constraint conditions were presented. Cornet and Laroque [8] introduced the properties of perturbed solutions in some mathematical programming with non-smooth objective function or constrained functions. On the whole, the aforementioned researches on steady-state traffic assignment by the general sensitivity analysis results, whose disadvantage is either of higher dimensions, or is solved by some heuristic algorithms. Hereon, we try to construct a new dimension-reduced method: the sensitivity analysis results of original perturbed problem are approximated by the solution of equivalent unconstrained problem based on the relation between the Lagrange function and logarithmic barrier function combined with a Courant quadratic penalty term and the improved sensitivity results about some non-smooth nonlinear programming in Section 2. Then we apply the results to SUEA model and give the restricted proof in Section 3 and a numerical example in Section 4. Finally, the conclusions and acknowledgements are summarized, respectively, in Sections 5 and 6.

## 2. Approximation of the first-order sensitivity information using a penalty method algorithm

Consider general parametric nonlinear programming problem defined as:

$$P(\varepsilon)\qquad \begin{aligned}&\min_x f(x,\varepsilon),\\ &\text{s.t.}\begin{cases} g_i(x,\varepsilon)\geqslant 0 & (i=1,2,\ldots,m),\\ h_j(x,\varepsilon)=0 & (j=1,2,\ldots,p),\end{cases}\end{aligned} \tag{2.1}$$

where $x\in\mathbb{R}^n$ and $\varepsilon$ is a column parameter vector in $\mathbb{R}^k$, and $f,g_i,h_j:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}$.

The Lagrangian function and logarithmic barrier function combined with a Courant quadratic penalty term of $P(\varepsilon)$ are defined as:

$$\begin{aligned}L(x,\pi\,\mu,\varepsilon)&=f(x,\varepsilon)-\sum_{i=1}^{m}\pi_i g_i(x,\varepsilon)+\sum_{j=i}^{p}\mu_j h_j(x,\varepsilon),\\ W(x,\varepsilon,r)&=f(x,\varepsilon)-r\sum_{i=1}^{m}\ln g_i(x,\varepsilon)+\frac{1}{2r}\sum_{j=1}^{p}h_j^2(x,\varepsilon),\end{aligned} \tag{2.2}$$

where $\pi_i,\mu_j$ are the associated Lagrange multipliers corresponding to the constraint conditions and $r$ is a positive real parameter.

Firstly, we give the following Lemma in Fiacco [1] and Theorem without proof mentioned in Cornet [8].

**Lemma 1.** (Second-order sufficient conditions for a local isolated minimizing point of problem $P(0)$). *If the functions defining problem $P(0)$ are twice continuously differentiable in a neighborhood of $x^*$, then $x^*$ is a local isolated minimizing point of problem $P(0)$ if there exist Lagrange multipliers $\pi^*$ and $\mu^*$ such that the first-order Kuhn–Tucker conditions hold, and further if $y^T\nabla^2 L(x^*,\pi^*,\mu^*,0)y>0$ for all $y\neq 0$ such that:*

$$\begin{cases} y^T\nabla g_i(x^*,0)\geqslant 0, & \text{for all } i,\ \text{where } g_i(x^*,0)=0,\\ y^T\nabla g_i(x^*,0)=0, & \text{for all } i,\ \text{where } \pi_i^*>0,\\ y^T\nabla h_j(x^*,0)=0, & j=1,2,\ldots,p.\end{cases}$$

**Theorem 1** (The perturbed local solution of $P(\varepsilon)$ and its property). *If $P(\varepsilon)$ satisfies*

(1) *$U$ is an open subset of $\mathbb{R}^n$ containing $x^*$, $V$ is an open subset of $\mathbb{R}^k$ containing $\varepsilon^*=0$, for all $\varepsilon$ in $V$, the mappings $f(\cdot,\varepsilon):U\to\mathbb{R},g_i(\cdot,\varepsilon):U\to\mathbb{R},h_j(\cdot,\varepsilon):U\to\mathbb{R}$ are twice Fréchet differentiable, and the mappings $D^2f(\cdot,\cdot):U\times V\to\mathbb{R}^{n^2}$, $D^2g_i(\cdot,\cdot):U\times V\to\mathbb{R}^{n^2},D^2h_j(\cdot,\cdot):U\times V\to\mathbb{R}^{n^2}$ are continuous at $(x^*,\varepsilon^*)$, and $g_i(\cdot,\cdot):U\times V\to\mathbb{R},\nabla g_i(\cdot,\cdot):U\times V\to\mathbb{R}^n,h_j(\cdot,\cdot):U\times V\to\mathbb{R},\nabla h_j(\cdot,\cdot):U\times V\to\mathbb{R}^n,\nabla f(\cdot,\cdot):U\times V\to\mathbb{R}^n$ are Lipschitzian continuously;*