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# Sensitivity analysis for frictional contact problems in the augmented Lagrangian formulation

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ABSTRACT

Direct differentiation method of sensitivity analysis is developed for frictional contact problems. As a result of the augmented Lagrangian treatment of contact constraints, the direct problem is solved simultaneously for the displacements and Lagrange multipliers using the Newton method. The main purpose of the paper is to show that this formulation of the augmented Lagrangian method is particularly suitable for sensitivity analysis because the direct differentiation method leads to a non-iterative exact sensitivity problem to be solved at each time increment. The approach is applied to a general class of three-dimensional frictional contact problems, and numerical examples are provided involving large deformations, multibody contact interactions, and contact smoothing techniques.



## 1. Introduction

Sensitivity analysis provides quantitative information on the variation of the response of a system, e.g. a mechanical system, related to a variation of parameters, e.g. material or shape parameters, on which the system depends. Response sensitivity is thus essentially a derivative of the response with respect to these parameters, however, the dependence is implicit—through the governing equations of the problem. Typically, the need for response sensitivities arises in optimization and inverse problems whenever gradient-based minimization algorithms are used, however, numerous other applications appear in the engineering practice, e.g. imperfection sensitivity analysis, error analysis, and others.

Efficient analytical methods of sensitivity analysis are already available for a variety of problems, see the monographs of Kleiber et al. [16], Choi and Kim [6], Kowalczyk [22]. In particular, this concerns the path-dependent problems of elasto-plasticity, cf. Tsay and Arora [42], Kleiber [15], Michaleris et al. [28].

Frictional contact problems, being the main concern of this work, also belong to the class of path-dependent problems. Continuum formulations for the most general case of three-dimensional multibody, large deformation frictional contact problems have been developed by Laursen and Simo [25], Klarbring [14], Pietrzak and Curnier [31,32]. In these formulations, an essential role is played by the contact constraints enforced on the potential contact surfaces, and several approaches have been developed for the treatment of these constraints and for the discretization of the contact interaction terms, see the recent monographs of Laursen [24] and Wriggers [43].

The penalty method is the simplest and apparently the most widely used method of enforcing contact constraints. It leads to a purely displacement formulation and straightforward implementation, however the constraints cannot be enforced exactly. Furthermore, ill-conditioning appears when the penalty parameter is increased in order to improve satisfaction of the constraints. Recently, the augmented Lagrangian method has become a popular alternative, free of the main drawbacks of the penalty method. There are two solution schemes commonly used in the context of the augmented Lagrangian method. The so-called Uzawa method, which seems to be more popular, combines the augmented Lagrangian regularization with a first-order update of Lagrange multipliers, cf. Simo and Laursen [36]. Alternatively, a Newton-like solution scheme can be applied to solve the saddle-point problem simultaneously for the displacements and Lagrange multipliers, cf. Alart and Curnier [2], Pietrzak and Curnier [31,32]. The latter approach is adopted in this work, however, the penalty and Uzawa methods are also discussed from the point of view of sensitivity analysis.

The most widely used contact discretization scheme is the node-to-segment approach. In order to avoid convergence problems caused by discontinuity of the normal vector, which is characteristic for the simple node-to-segment implementations, contact smoothing techniques are often introduced, e.g. Pietrzak [31], Puso and Laursen [33], Krstulović-Opara et al. [23]. This approach is used in the present work. An alternative approach, employing the mortar method and leading to a segment-to-segment strategy, has recently been developed, cf. Puso and Laursen [34].

Several developments of sensitivity analysis for frictional contact problems have been reported in the literature. If the penalty approach

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is adopted, then the general structure of the frictional contact problem is similar to that of the elasto-plasticity, so that the respective formalism of sensitivity analysis by the direct differentiation method can be directly applied, cf. Kim et al. [12,13], Stupkiewicz et al. [40]. In the case of the augmented Lagrangian method with the Uzawa-type algorithm, the exact sensitivity analysis requires iterations corresponding to the iterative update scheme for Lagrange multipliers. In order to avoid this, Srikanth and Zabaras [37] have introduced an approximate non-iterative sensitivity problem in which oversized penalties are used. Sensitivity analysis methods specialized for frictionless contact problems have been developed, e.g., by Bendsoe et al. [4], Haslinger et al. [10], Fancello et al. [8], Tardieu and Constantinescu [41], Hilding et al. [11]. On the other hand, approaches applicable to optimization of metal forming processes with a rigid-plastic material behaviour have been proposed, e.g., by Kleiber and Sosnowski [17], Antunez and Kleiber [3], and Fourment et al. [9].

In this work, the direct differentiation method (DDM) of sensitivity analysis is applied to frictional contact problems in the augmented Lagrangian formulation. In particular, we show that the augmented Lagrangian treatment of contact constraints, proposed by Alart and Curnier [2] and Pietrzak and Curnier [31,32], is particularly suitable for sensitivity analysis because it leads to a non-iterative exact sensitivity problem at each time step. In order to illustrate that, sensitivity analysis is developed for a general class of three-dimensional contact problems including friction, multibody contact interaction and contact smoothing. Preliminary results concerning sensitivity analysis for the augmented Lagrangian formulation and for contact smoothing techniques have been published in conference papers [39] and [26], respectively.

The paper is organized as follows. A framework for the sensitivity analysis for path-dependent problems is presented in Section 2. The direct problem is introduced in a general discretized form, and then the direct differentiation method is used to derive the corresponding sensitivity problem. In Section 3 we show that this general framework applies also to the contact problems. Contact kinematics and contact constraints are specified in Section 3.1. Detailed discussion of the penalty and augmented Lagrangian methods for a simplified case of elastic frictionless problems is provided in Section 3.2. Finally, the general frictional contact problem is formulated in Section 3.3. Remarks concerning the finite element implementation are provided in Section 3.4, and illustrative numerical examples are given in Section 4.

## 2. Sensitivity analysis for path-dependent problems

A general framework for the sensitivity analysis of path-dependent problems is introduced in this section. The exposition follows the one presented by Michaleris et al. [28] for the case of *transient coupled systems* in the terminology of [28]. This class of problems includes elasto-plasticity and, as shown in Section 3, frictional contact, the latter being of the main interest in this work.

### 2.1. Direct problem

Consider the following general discrete path-dependent problem in a residual form resulting, for instance, from the finite element discretization and time integration of the respective variational principle (e.g. the virtual work principle),

$$\mathbf{R}_n(\mathbf{p}_n, \mathbf{h}_n, \mathbf{p}_{n-1}, \mathbf{h}_{n-1}) = \mathbf{0}, \tag{1}$$

$$\mathbf{Q}_n(\mathbf{p}_n, \mathbf{h}_n, \mathbf{p}_{n-1}, \mathbf{h}_{n-1}) = \mathbf{0}, \tag{2}$$

where $(\mathbf{p}_n, \mathbf{h}_n)$ are the global vectors of unknowns at the *current* time increment $t = t_n$ and $(\mathbf{p}_{n-1}, \mathbf{h}_{n-1})$ are the known values at the *previous* time increment, $t = t_{n-1}$. The problem is path-dependent hence the dependence of the residuals $\mathbf{R}_n$ and $\mathbf{Q}_n$ on the solution at the previous time increment as in Eqs. (1)–(2).

Typically, vector $\mathbf{p}_n$ would be the global vector of nodal displacements, and the residual equation $\mathbf{R}_n = \mathbf{0}$ would then correspond to the equilibrium of nodal forces. However, depending on the problem and its formulation, $\mathbf{p}_n$ may in general contain arbitrary global unknowns (e.g. Lagrange multipliers enforcing contact constrains, as discussed in Section **3**). At the same time, $\mathbf{h}_n$ assembles the state variables which are defined locally at each element or integration point. Accordingly, it is more efficient to formulate the problem as a coupled one, as in Eqs. (1)–(2), rather than to solve for $(\mathbf{p}_n, \mathbf{h}_n)$ monolithically. This is because the local residuals assembled in $\mathbf{Q}_n = \mathbf{0}$ can be independently solved at each element or integration point. The resulting iteration–subiteration procedure is presented below.

Eqs. (1) and (2) are solved by assuming that $\mathbf{h}_n$ is a function of $\mathbf{p}_n$,

$$\mathbf{R}_n(\mathbf{p}_n, \mathbf{h}_n(\mathbf{p}_n)) = \mathbf{0}, \tag{3}$$

$$\mathbf{Q}_n(\mathbf{p}_n, \mathbf{h}_n(\mathbf{p}_n)) = \mathbf{0}, \tag{4}$$

where the dependence on $\mathbf{p}_{n-1}$ and $\mathbf{h}_{n-1}$ is temporarily suppressed because these values are fixed and known at the current time increment.

The solution is obtained by applying the Newton method in two nested iterative loops. Given the current estimate $\mathbf{p}_n^I$, Eq. (4) is iteratively solved for $\mathbf{h}_n$ in the inner Newton loop. Linearization of Eq. (4) with respect to $\mathbf{h}_n^J$ at fixed $\mathbf{p}_n^I$ provides a linear equation for the correction $\Delta\mathbf{h}_n^J$, namely

$$\frac{\partial \mathbf{Q}_n}{\partial \mathbf{h}_n}\Delta\mathbf{h}_n^J = -\mathbf{Q}_n\left(\mathbf{p}_n^I, \mathbf{h}_n^J\right). \tag{5}$$

Once $\Delta\mathbf{h}_n^J$ is computed from Eq. (5), $\mathbf{h}_n$ is updated according to

$$\mathbf{h}_n^{J+1} = \mathbf{h}_n^J + \Delta\mathbf{h}_n^J, \tag{6}$$

and the procedure is repeated until $\mathbf{h}_n^J$ converges to the solution $\mathbf{h}_n(\mathbf{p}_n^I)$. This solution implicitly depends on $\mathbf{p}_n^I$ through $\mathbf{Q}_n = \mathbf{0}$, cf. Eq. (4). The derivative of this dependence, needed below, can be obtained by differentiating Eq. (4) with respect to $\mathbf{p}_n$, namely

$$\frac{\partial \mathbf{Q}_n}{\partial \mathbf{h}_n}\frac{\mathrm{D}\mathbf{h}_n}{\mathrm{D}\mathbf{p}_n} = -\frac{\partial \mathbf{Q}_n}{\partial \mathbf{p}_n}, \tag{7}$$

where $\partial\mathbf{Q}_n/\partial\mathbf{h}_n$ is the dependent tangent operator appearing also in the iterative scheme (5)–(6). Here and below the derivatives of implicit dependencies are denoted by $\mathrm{D}(\cdot)/\mathrm{D}(\cdot)$.

Now, the outer Newton loop is associated with the iterative solution of $\mathbf{R}_n = \mathbf{0}$. Linearization of Eq. (3) yields the equation for the correction $\Delta\mathbf{p}_n^I$,

$$\left[\frac{\partial \mathbf{R}_n}{\partial \mathbf{p}_n} + \frac{\partial \mathbf{R}_n}{\partial \mathbf{h}_n}\frac{\mathrm{D}\mathbf{h}_n}{\mathrm{D}\mathbf{p}_n}\right]\Delta\mathbf{p}_n^I = -\mathbf{R}_n\left(\mathbf{p}_n^I, \mathbf{h}_n\left(\mathbf{p}_n^I\right)\right). \tag{8}$$

The term $\mathrm{D}\mathbf{h}_n/\mathrm{D}\mathbf{p}_n$ has already been determined in Eq. (7), so that the linear equation for the correction $\Delta\mathbf{p}_n^I$ finally takes the form

$$\left[\frac{\partial \mathbf{R}_n}{\partial \mathbf{p}_n} - \frac{\partial \mathbf{R}_n}{\partial \mathbf{h}_n}\left(\frac{\partial \mathbf{Q}_n}{\partial \mathbf{h}_n}\right)^{-1}\frac{\partial \mathbf{Q}_n}{\partial \mathbf{p}_n}\right]\Delta\mathbf{p}_n^I = -\mathbf{R}_n\left(\mathbf{p}_n^I, \mathbf{h}_n\left(\mathbf{p}_n^I\right)\right), \tag{9}$$

where the term in the square brackets is the independent tangent operator, also called the consistent or algorithmic tangent operator. Here and below, we only consider regular states in which the tangent matrix is not singular (treatment of critical or bifurcation points requires separate analysis, e.g., [29]). Upon solving Eq. (9) for $\Delta\mathbf{p}_n^I$, the response $\mathbf{p}_n$ is updated according to

$$\mathbf{p}_n^{I+1} = \mathbf{p}_n^I + \Delta\mathbf{p}_n^I, \tag{10}$$