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On the interdependency of primary and initial secondary equilibrium paths in sensitivity analysis of elastic structures

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ABSTRACT

The scientific motivation for this paper is lack of clarity about the interdependency of primary and initial secondary equilibrium paths in the frame of sensitivity analysis of elastic structures. The investigation of this interdependency comprises of the following four cases: (1) nonlinear primary path, nonlinear stability problem, (2) linear primary path, nonlinear stability problem, (3) nonlinear primary path, linear stability problem, and (4) linear primary path, linear stability problem. The consistently linearized eigenproblem is used for differentiation of two classes of nonlinear stability problems with markedly different characteristics of both the prebuckling and the postbuckling behavior. For one of them, e.g. zero-stiffness postbuckling is impossible. For the other one, which is restricted to a prebuckling regime with axial deformations only, sensitivity analysis of the initial postbuckling behavior either exhibits its continuous improvement or its continuous deterioration, depending on whether the bifurcation point diverges from or converges to the snap-through point. In other words, a monotonic variation of the design parameter cannot result in a non-monotonic change of the initial postbuckling behavior. The practical motivation for this work is to explore the mechanical reasons for qualitatively different modes of transition from imperfection sensitivity to insensitivity in the course of sensitivity analysis for the purpose of improving the postbuckling behavior of structures by means of minor design changes. Results from a numerical investigation corroborate the theoretical findings.

1. Introduction

Usually, the main goal of sensitivity analysis of elastic structures susceptible to buckling is to increase their stability limits through variation of suitable design variables. However, the resulting improvement in prebuckling behavior may be accompanied by a significant deterioration of the postbuckling behavior (Fig. 1). In Fig. 1, λ is a dimensionless load factor by which the reference load vector $\bar{\mathbf{P}}$, in the frame of the Finite Element Method (FEM), is multiplied, and u denotes a component of the vector of nodal displacements $\mathbf{q}$; S denotes the bifurcation point and D and D' stand for the snap-through point of the perfect and the imperfect structure, respectively. In Fig. 1(b), the bifurcation point coincides with the snap-through point. This is called hilltop buckling.

Alternatively, the improvement of the prebuckling behavior may be accompanied by a significant improvement of the postbuckling behavior (Fig. 2). The secondary equilibrium path in Fig. 2(b) represents zero-stiffness postbuckling. It is bound up with a particularly favorable form of transition from imperfection sensitivity (Fig. 2(a)) to imperfection insensitivity (Fig. 2(b) and (c)).

Figs. 1 and 2 represent two extreme cases of results from sensitivity analysis of elastic structures with special emphasis on buckling and postbuckling. Bochenek [1] has performed such analysis for the purpose of optimizing the postbuckling behavior of structures. Mang et al. [2] presented theoretical results on the conversion from imperfection-sensitive into imperfection-insensitive elastic structures. In a comprehensive numerical investigation Schranz et al. [3] verified these results. Hilltop buckling, representing one of many details of such an investigation, was studied by Fujii and Noguchi [4], Ohsaki and Ikeda [5], and Mang et al. [6]. Tarnai [7] pioneered research on zero-stiffness postbuckling which is another one of these details. Recently, Mang et al. [8] gave a mathematical proof of the predictability of zero-stiffness postbuckling. Jia et al. [9] have shown that zero-stiffness postbuckling is imperfection insensitive.

Figs. 1 and 2 are examples of the interdependency of primary and secondary equilibrium paths of elastic structures. So far, a systematic investigation of this interdependency is lacking. This has been the motivation for the present work. Emphasis will be laid on an important borderline case that is associated with the end of imperfection sensitivity and the beginning of imperfection insensitivity, respectively, in sensitivity analysis of the initial postbuckling behavior. For special forms of this case, the sign of

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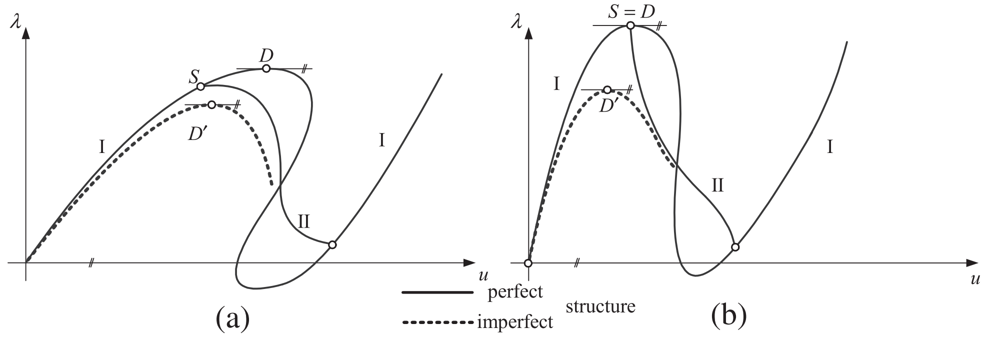

Fig. 1. Sketches illustrating the improvement of the prebuckling behavior accompanied by a deterioration of the postbuckling behavior of a perfect structure, and its influence on an imperfect structure (a) at the start and (b) at the end of the variation of a design parameter (hilltop buckling). (I: primary (prebuckling) path, II: secondary (postbuckling) path).

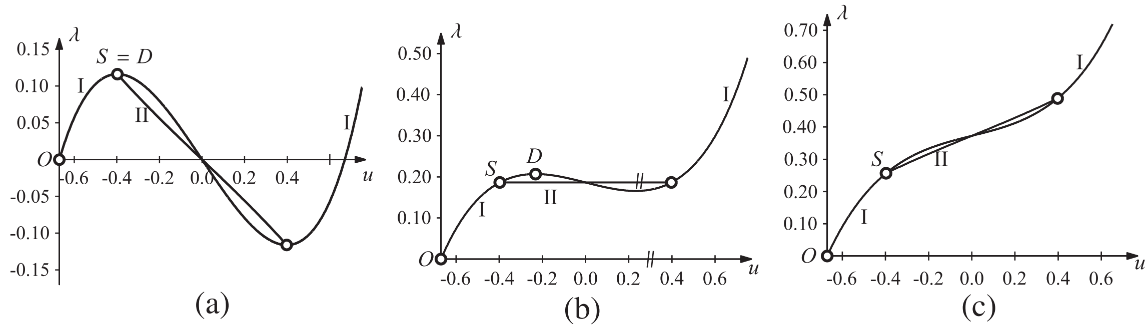

Fig. 2. Improvement of the prebuckling behavior accompanied by improvement of the postbuckling behavior of a perfect structure (a) at the start (hilltop buckling), (b) at zero-stiffness postbuckling, and (c) at the end of the variation of a design parameter. (I: primary (prebuckling) path, II: secondary (postbuckling) path).

a particular mathematical expression allows identification of essential properties of the primary and the secondary equilibrium path. This expression is an ingredient of a mathematical formula that was derived by Mang et al. [2] with the help of Koiter's initial postbuckling analysis [10]. Hence, in principle, the investigation of the postbuckling behavior is restricted to the vicinity of the bifurcation point. Nevertheless, the aforementioned predictability of zero-stiffness postbuckling allows a statement about the entire secondary equilibrium path based only on information that is available at the stability limit.

Section 2 is devoted to primary equilibrium paths. Four different cases will be investigated: (1) nonlinear primary path, nonlinear stability problem, (2) linear primary path, nonlinear stability problem, (3) nonlinear primary path, linear stability problem, and (4) linear primary path, linear stability problem. This investigation aims at clearing up the classical misunderstanding of a mutual conditionality of linear prebuckling paths and linear stability problems that was addressed by Steinboeck and Mang [11]. In Section 3, general expressions for sensitivity analysis of the initial postbuckling path will be presented. The so-called consistently linearized eigenproblem, proposed by Helnwein [12], will be used in Section 4 for differentiation between two classes of nonlinear stability problems as regards sensitivity analysis of the initial postbuckling behavior. In Section 5, two important special cases of postbuckling behavior will be discussed. They are (a) hilltop buckling and (b) zero-stiffness postbuckling. Section 6 contains a summary of findings from the theoretical investigation of the four cases discussed in Section 2. Section 7 is devoted to numerical verification of theoretical results. In Section 8, the conclusions drawn from this work will be presented. Appendix A contains some of the mathematical expressions of quantities that are used in this work.

In accord with the goals of this work, it is restricted to perfect structures (with one exception) and to distinct bifurcation points.

2. Primary equilibrium paths

2.1. Nonlinear primary path, nonlinear stability problem

In the frame of the FEM, the infinitesimally incremental form of the equilibrium equations is given as [13]

$$\widetilde{\mathbf{K}}_T \cdot \mathrm{d}\mathbf{q} = \mathrm{d}\lambda \bar{\mathbf{P}}, \tag{1}$$

where $\widetilde{\mathbf{K}}_T$ indicates specialization of the tangent stiffness matrix $\mathbf{K}_T$ for the primary path. Rewriting (1) as

$$\widetilde{\mathbf{K}}_T \cdot \mathbf{q}_{,\lambda} = \bar{\mathbf{P}} \tag{2}$$

with

$$\mathbf{q}_{,\lambda} := \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\lambda} \tag{3}$$

and computing the first derivative of (2) with respect to λ, yields

$$\widetilde{\mathbf{K}}_{T,\lambda} \cdot \mathbf{q}_{,\lambda} + \widetilde{\mathbf{K}}_T \cdot \mathbf{q}_{,\lambda\lambda} = \mathbf{0}. \tag{4}$$

The eigenvector of $\widetilde{\mathbf{K}}_T$, which refers to the bifurcation point related to loss of stability of the primary equilibrium path, representing the buckling mode, is denoted as $\boldsymbol{v}_1$. Premultiplication of (1) by the transpose of $\boldsymbol{v}_1$ yields

$$\boldsymbol{v}_1^T \cdot \widetilde{\mathbf{K}}_T \cdot \mathrm{d}\mathbf{q} = \mathrm{d}\lambda \boldsymbol{v}_1^T \cdot \bar{\mathbf{P}}. \tag{5}$$