Computers and Mathematics with Applications 64 (2012) 190–200

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Sensitivity analysis for a structured juvenile–adult model

Azmy S. Ackleh *, Keng Deng, Xing Yang

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, United States

ARTICLE INFO

Keywords:
Structured juvenile–adult model
Sensitivity equations
Finite difference approximation

ABSTRACT

In this paper, we consider a model which describes the dynamics of an amphibian population where individuals are divided into juveniles and adults. We derive sensitivity partial differential equations for the sensitivities of the solution with respect to the reproduction and mortality rates for adults. We also present numerical results to show the application of these equations to an amphibian population of green tree frogs (*Hyla cinerea*).

1. Introduction

In this paper, we consider the following structured juvenile–adult population model:

$$
\begin{aligned}
&J_t(a,t) + J_a(a,t) + \nu(a)J(a,t) = 0, && (a,t) \in (0,\bar{a}) \times (0,T),\\
&A_t(x,t) + (g(x)A(x,t))_x + \mu(\varphi(t))A(x,t) = 0, && (x,t) \in (\underline{x},\bar{x}) \times (0,T),\\
&J(0,t) = \int_{\underline{x}}^{\bar{x}} \beta(\varphi(t))A(x,t)dx, && t \in (0,T),\\
&g(\underline{x})A(\underline{x},t) = J(\bar{a},t), && t \in (0,T),\\
&J(a,0) = J_0(a), && a \in [0,\bar{a}],\\
&A(x,0) = A_0(x), && x \in [\underline{x},\bar{x}],
\end{aligned}
\tag{1.1}
$$

where $J(a,t)$ and $A(x,t)$ denote the density of juveniles of age a and adults of size x at time t, respectively, $\bar{a}$ denotes the age at which juveniles metamorphose into adults of minimum size $\underline{x}$, and $\bar{x}$ denotes the maximum size of adults. The function $\varphi(t) = \int_{\underline{x}}^{\bar{x}} A(x,t)dx$ is the total population of adults. The parameters ν and μ are the mortality rates for juveniles and adults, respectively. The functions g and β are the growth and reproduction rates for adults, respectively. Motivated by an amphibian population of green tree frogs (*Hyla cinerea*), we recently developed such a model in [1]. We assumed that juveniles live in an environment with abundant resources and thus do not compete, while adults live in an environment with limiting resource and thus competition between them takes place. We then established existence–uniqueness results and discussed the long-time behavior of the solution of the model via a comparison principle.

Our objective here is to conduct sensitivity analysis for model (1.1). The importance of sensitivity equations has long been recognized as they provide a measure of model response (output) to variation in the underlying model parameters (e.g., see [2,3] and the references therein). The derivation of sensitivity equations for discrete structured population models (matrix models) has received a great deal of attention in the past few decades (see [2] and the many references therein).

* Corresponding author.
E-mail address: ackleh@louisiana.edu (A.S. Ackleh).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.12.053

However, little work has been done on the derivation of sensitivity equations for continuous structured population models which are of Mckendrick–von Foerster partial differential equations type. Such equations are useful for computing variances of estimated model parameters from observation data (e.g., see, [4–7]). Our motivation comes from paper [8], wherein sensitivity partial differential equations were derived for the following linear size-structured population model:

$$\begin{aligned} &u_t(x,t) + (g(x)u(x,t))_x + \mu(x)u(x,t) = 0, && (x,t) \in (0,\bar{x}) \times (0,T), \\ &g(0)u(0,t) = \int_0^{\bar{x}} \beta(x)u(x,t)dx, && t \in (0,T), \\ &u(x,0) = u_0(x), && x \in [0,\bar{x}]. \end{aligned} \tag{1.2}$$

There are two main differences between models (1.2) and (1.1). First, (1.2) is a single equation model, but (1.1) is a system of coupled equations. Second, the vital rates in (1.2) are linear, but the reproduction and mortality rates for adults in (1.1) are dependent on the total population of adults. Due to the different structure of model (1.1), the situation becomes more complicated, and certain techniques used for (1.2) seem not applicable to (1.1).

The paper is organized as follows. In Section 2, we establish an existence result for directional derivatives with respect to parameters. In Section 3, we derive sensitivity partial differential equations for the sensitivities of the solution with respect to the reproduction and mortality rates for adults. In Section 4, we make numerical simulations to apply these equations to the population of green tree frogs.

2. Existence of the directional derivative

To carry out our sensitivity analysis, we assume that the parameters in (1.1) satisfy the following assumptions:

(A1) $g \in C^1[\underline{x}, \bar{x}]$. Furthermore, $g(x) > 0$ for $x \in [\underline{x}, \bar{x})$ and $g(\bar{x}) = 0$.
(A2) $\nu \in L^\infty(0, \bar{a})$ is nonnegative.
(A3) $\mu \in C^1[0, \infty)$ is nonnegative with $\mu' \geq 0$.
(A4) $\beta \in C^1[0, \infty)$ is nonnegative with $\beta' \leq 0$.
(A5) $J_0 \in L^\infty(0, \bar{a})$ is nonnegative.
(A6) $A_0 \in L^\infty(\underline{x}, \bar{x})$ is nonnegative.

We first introduce the solution representation for problem (1.1) via the method of characteristics. For the first equation in (1.1), the characteristic curves can be easily obtained. For the second equation in (1.1), the characteristic curves are given by

$$\begin{cases} \dfrac{d}{ds}t(s) = 1 \\ \dfrac{d}{ds}x(s) = g(x(s)). \end{cases} \tag{2.1}$$

Under assumption (A1), Eq. (2.1) has a unique solution for any initial point $(x(s_0), t(s_0))$. Parameterizing the characteristic curves with the variable t, then a characteristic curve passing through $(\hat{x}, \hat{t})$ is given by $(X(t; \hat{x}, \hat{t}), t)$, where X satisfies

$$\frac{d}{dt}X(t;\hat{x},\hat{t}) = g(X(t;\hat{x},\hat{t}))$$

and $X(\hat{t}; \hat{x}, \hat{t}) = \hat{x}$. By (A1) the function X is strictly increasing, and therefore a unique inverse function $\Gamma(x; \hat{x}, \hat{t})$ exists. Let $G(x) = \Gamma(x; \underline{x}, 0)$, then $(x, G(x))$ represents the characteristic curve passing through $(\underline{x}, 0)$, and this curve divides the (x, t)-plane into two parts. Hence, the solution of (1.1) can be represented as follows:

$$J(a,t) = J_0(a-t)\exp\left(-\int_{a-t}^{a} \nu(\sigma)d\sigma\right) \quad \text{if } t \leq a, \tag{2.2}$$

$$J(a,t) = \beta(\varphi(t-a))\varphi(t-a)\exp\left(-\int_0^{a} \nu(\sigma)d\sigma\right) \quad \text{if } t > a, \tag{2.3}$$

$$A(x,t) = A_0(X(0;x,t))\exp\left\{-\int_0^t [g_x(X(\tau;x,t)) + \mu(\varphi(\tau))]d\tau\right\} \quad \text{if } t \leq G(x), \tag{2.4}$$

$$A(x,t) = \frac{J(\bar{a}, \Gamma(\underline{x};x,t))}{g(\underline{x})}\exp\left\{-\int_{\Gamma(\underline{x};x,t)}^t [g_x(X(\tau;x,t)) + \mu(\varphi(\tau))]d\tau\right\} \quad \text{if } t > G(x). \tag{2.5}$$

Integrating (2.4)–(2.5) with respect to x, we obtain an integral representation for $\varphi(t)$:

$$\varphi(t) = \int_0^t J(\bar{a},\tau)\exp\left(-\int_\tau^t \mu(\varphi(s))ds\right)d\tau + \int_{\underline{x}}^{\bar{x}} A_0(\xi)\exp\left(-\int_0^t \mu(\varphi(s))ds\right)d\xi. \tag{2.6}$$