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Sensitivity analysis of CRA based controllers in fractional order systems

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ABSTRACT

This paper focuses on robust performance analysis of a closed loop fractional order system through a sensitivity approach. The characteristic ratio assignment method is selected to attain a desired closed loop transient response. Then, we compute the sensitivity of such a desired transfer function with respect to its characteristic ratio and we explore its specifications. The relation between the coefficient diagram shape and the relative stability of the closed loop system is discussed. Also, the closed loop poles variations due to the changes in the characteristic ratios are investigated. Finally, we study a pseudo second order process to verify the robust performance of the characteristic ratio assignment approach with RST control structure.

1. Introduction

Benefits of employing fractional order operators in modeling, identification, and control, encourage scientists to investigate different fields of fractional calculus [1–4]. Most of the physical processes could be represented better with fractional order models, especially those including viscoelasticity, diffusion, and thermoelasticity [5–8]. Designing appropriate controllers for fractional order models is a main research subject in this regard. A lot of control strategies have been proposed in the literature to improve the performance of a fractional order system [9–15]. Among them, characteristic ratio assignment (CRA) method is a novel analytical approach to control the transient response of such systems [15]. In this method, the characteristic ratios which could be represented in terms of characteristic equation coefficients are assigned to gain a non overshooting step response. The speed adjustment of the transient response could be independently performed by selecting generalized time constant in accordance with the time scaling property. The change in the generalized time constant only scales the transient response without any effect on its damping or overshoot.

Designing a robust control system which is less sensitive to changes in the process parameters is one of the main goals in control theory. The sensitivity of such a system would be low with respect to perturbation in the process parameters. Thus, the sensitivity analysis of a control structure could help to investigate the robustness of its closed loop system. Sensitivity analysis of a CRA based fractional order controller is the main contribution of this paper. Some useful relations to compute the sensitivity of a closed loop transfer function and its poles due to variations in the characteristic ratios and characteristic equation coefficients are presented. Through this analytical approach, some important qualitative results are derived which could help to design a robust control system. Coefficient diagram for the proposed characteristic equation is plotted and its relation to the relative stability is illustrated. The sensitivity of the closed loop dominant poles to changes in the process parameters is discussed, as well. To verify the obtained results in a

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commonly used closed loop system, a pseudo second order process with uncertain parameters is considered. Based on the CRA method, an RST control structure is build to attain the desired closed loop transfer function. The robustness of the proposed controller is checked through the sensitivity analysis and results are confirmed based on computer simulations of the controller.

This paper is organized as follows. Section 2 gives a review on the CRA method and its properties for fractional order systems. Some relations to calculate the sensitivity of an all-pole fractional order system to its characteristic ratios are given in Section 3. Section 4 deals with the sensitivity analysis of a desired closed loop transfer function obtained through the CRA method to its characteristic ratios variation. The general shape of the coefficient diagram for the proposed characteristic ratio pattern and its relation to the relative stability is discussed. Robust performance verification of a closed loop system under parametric uncertainties in a case study process is given in Section 5. Section 6 concludes the paper.

2. Characteristic ratio in fractional order systems

The memory contained in the fractional order derivative makes it different from the ordinary derivative. The infinite dimension of fractional order systems is the result of the long memory principle. Analytical computation of the fractional derivative is a complex issue due to the long memory property. To overcome this limitation, different approximation methods have been proposed to replace a fractional order derivative [16–18]. In this paper FOTF toolbox introduced in [18] has been employed to simulate a fractional order system.

A fractional order system with derivative terms that are integer multiples of a common factor (called commensurate order) is named as a commensurate system [2]. The following transfer function represents an all-pole commensurate fractional order system.

$$G(s)=\frac{a_0}{p(s)},\quad p(s)=a_n s^{n\nu}+a_{n-1}s^{(n-1)\nu}+\cdots+a_1 s^{\nu}+a_0. \quad (1)$$

The characteristic ratio assignment was firstly introduced to control the transient response of ordinary systems [19–22]. This method is based on the separation of time and amplitude specifications of a closed loop step response. The principles used in the CRA method could be similarly generalized for fractional order systems [15].

The transient response of system (1) could be uniquely described by characteristic ratios ($\alpha_k, k=1,\ldots,n-1$) and generalized time constant (τ) which are defined as follows [15]:

$$\alpha_i=\frac{a_i^2}{a_{i-1}a_{i+1}}, i=1,\ldots,n-1,\quad \tau=\left(\frac{a_1}{a_0}\right)^{1/\nu}. \quad (2)$$

The amplitude properties (such as maximum overshoot) of the step response for system (1) are determined by α_i while the time specifications (such as rise time, settling time, and alike) are adjusted by τ. In other words, systems with the same characteristic ratios have similar damping in their step responses and different speeds depending on their generalized time constant.

The characteristic equation coefficients could be uniquely determined in terms of the characteristic ratios and the generalized time constant as follows [15]:

$$a_1=\tau^{\nu}a_0,\quad a_i=\frac{a_0\tau^{i\nu}}{\alpha_{i-1}\alpha_{i-2}^2\alpha_{i-3}^3\cdots\alpha_1^{i-1}},\quad i=2,3,\ldots,n. \quad (3)$$

Characteristic ratio assignment to get a step response with a minimum overshoot (say 2%) for the case $0.5<\nu\le 1$ has been presented in [15]. The proposed pattern for the characteristic ratios has the following form

$$\alpha_i=\begin{cases}-2\beta\cos(\pi\nu), & \text{if}: i=2k+1,\\ \frac{-2}{\beta\cos(\pi\nu)}, & \text{if}: i=2k,\end{cases} \quad (4)$$

where $k\in$ naturals; and β is an adjustable parameter which could be tuned to obtain a predefined overshoot. For example to reach a 2% overshoot, the following relation has been proposed for β in [15]

$$\beta=1.254\nu^{4.717}-0.05652. \quad (5)$$

The characteristic ratio pattern in (4) is called as ‘‘alternative characteristic ratio pattern’’ in the remainder of this paper. Analysis of systems designed with this alternative pattern from the sensitivity point of view is the main subject of this paper.

3. Sensitivity of a fractional order system to its characteristic ratios

In [23], the sensitivity of the transient response to its characteristic ratios has been studied. This section focuses on the sensitivity analysis of an all-pole fractional order system. Some useful relations are presented to study the sensitivity of a fractional order system to its characteristic ratios.

Let denote the sensitivity of transfer function in (1) to i-th characteristic ratio (α_i) by $S_{\alpha_i}^G$ and to coefficient a_j, $j=0,\ldots,n$ by $S_{a_j}^G$, and the sensitivity of a_j to characteristic ratio α_i by $S_{\alpha_i}^{a_j}$. Then the following relation holds

$$S_{\alpha_i}^G=\sum_{j=i+1}^{n}S_{a_j}^G S_{\alpha_i}^{a_j}. \quad (6)$$

According to (3), α_i affects a_j only for $j>i$. Thus $S_{\alpha_i}^{a_j}$ is zero for $j\le i$. This is why the summation index in (6) starts from $i+1$.

Calculation of $S_{\alpha_i}^G$ needs calculation of $S_{a_j}^G$ and $S_{\alpha_i}^{a_j}$. $S_{a_j}^G$ is calculated according to the definition of sensitivity.

$$S_{a_j}^G=\frac{\partial G}{\partial a_j}\frac{a_j}{G}=\frac{-a_0 s^{j\nu}}{p(s)^2}\frac{p(s)a_j}{a_0}=\frac{-a_j s^{j\nu}}{p(s)}. \quad (7)$$

Assuming $m=j-i$, $m>0$, the following relation is obtained

$$S_{\alpha_i}^{a_j}=\frac{\partial a_j}{\partial \alpha_i}\frac{\alpha_i}{a_j}=\frac{\partial a_j}{\partial \alpha_{j-m}}\frac{\alpha_{j-m}}{a_j}=\frac{a_0\tau^{j\nu}}{\prod_{l=1,l\neq m}^{j-1}\alpha_{j-l}^{l}}\frac{-m\alpha_{j-m}^{m-1}}{\alpha_{j-m}^{2m}}\frac{\alpha_{j-m}}{a_j}$$
$$=-m=-(j-i). \quad (8)$$

This means that the sensitivity of a coefficient to a characteristic ratio depends on the distance between their indices. Combination of (7) and (8) leads to the following relation that shows the sensitivity of the transfer function