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Frequency sensitivity analysis for beams carrying lumped masses with translational and rotary inertias

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ABSTRACT

The finite element method is applied to the sensitivity analysis of a natural frequency of a general beam carrying a lumped mass with both translational and rotary inertias. By virtue of the characteristics of the shape functions of a higher-order finite beam element of three degrees of freedom per node (namely, the translation, rotation and curvature), successfully formulated is a closed-form solution of the frequency sensitivity with respect to the attachment point of the mass. More importantly, by using the same element model, the first-order derivative of a natural frequency can be evaluated readily with the essential nodal displacements. Numerical results show that the sensitivity can be achieved with excellent precision.

For practical calculation of the frequency sensitivity, however, a further investigation is performed with use of the classical finite beam element of two degrees of freedom per node (i.e., the translation and rotation). Two approaches are provided for the curvature approximation at the mass-attached point. Comparison of numerical solutions from the uniform and linearly tapered beams illustrates that the frequency sensitivity can only be appropriately estimated in a more refined mesh scheme with the commonly-used beam element.

1. Introduction

The vibration problem involving beam-like structures with lumped (or concentrated) mass attachments arises frequently in civil structures, marine industry and mechanical engineering. In the past decades, the transverse vibration of a beam with one or more lumped masses attached in its span had been exhaustively studied due to its importance to industry [1–5]. In general, the vibration solution of a beam with a simple geometric section is gained by using analytical methods, such as the frequency determinant method or the Laplace transform method with consideration of the compatibility conditions at the attachment points of the lumped masses [1–5]. For a more complicated beam with the variable cross section, e.g., a generally tapered beam, acquisition of the natural frequency and the mode shape has to resort to the numerical approaches, like the finite element (FE) or Rayleigh–Ritz methods, etc. [6–8].

It is widely recognized that the frequency optimization is of great importance in the design of machines and structures subjected to dynamic loadings. In practical situations, moving a lumped mass around may be an interesting and effective way to control the dynamic properties of a structure if loaded with lumped masses, for example shifting a natural frequency away from the undesirable resonant region or setting up a desired resonant frequency gap [9]. Thus, frequency optimization of the beam-mass system with respect to the locations of lumped masses has always been a major concern of design engineers. For successively and efficiently performing the optimization procedure, the frequency sensitivity to the position of a mass attachment should first be obtained with good accuracy. With this sensitivity information, one can modify the mass position on purpose. Otherwise, adjustment of the mass-attached position has to be performed via a trial-and-error process. However, the related frequency sensitivity analysis has not yet been investigated sufficiently, especially with simultaneous inclusion of all the inertial effects of a lumped mass.

In many of the earlier literature, only the translational inertia of a lumped mass (or called a point mass) is concerned with in the vibration studies, whereas the rotary inertial effect, as a special feature of a lumped mass, is mostly neglected [3–5]. However, just as is pointed out in Refs. [1,2], the rotary inertia due to a lumped mass may impose a considerable influence upon the dynamic properties of a mass-loaded structure, especially for the higher-order vibration frequencies and modes. In some cases the rotary inertia may be much more important than the translational inertia of a lumped mass. For this reason, it should be involved in the position optimization of lumped masses for a beam- or plate-mass structure.

So far, there were only a few publications available on the frequency sensitivity formulation with respect to the location of a

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lumped mass in a system, and the analysis had primarily been focused on the transverse inertia (or magnitude). Specifically, Wang [10] applied a modal-based method to achieve a general expression of the frequency derivative of an Euler–Bernoulli beam by treating the impact of a lumped mass as an external excitation. Oguamanam et al. [11] performed the frequency sensitivity by means of the generalized Rayleigh's principle. Wang et al. [9] derived the sensitivity of a beam or plate based on the properties of the usual shape functions of the relevant finite elements. Although previous investigations have evidently shown that the rotary inertia of a lumped mass is also of great importance in the vibration analysis, it appears that the related sensitivity analysis to its position has not been appropriately explored in the open literature. This situation has been motivating the present author to investigate this problem with inclusion of both inertias of a lumped mass. More recently, an exact expression of the frequency sensitivity to the mass-attached position has been properly presented for a beam-mass system [12]. The sensitivity formulation is developed by virtually introducing additional degrees of freedom (DOFs) at the mass-attached point. However, in evaluation of the frequency sensitivity the analytical solution of the beam deflection has to be taken because the curvatures at the attachment position are typically required in the sensitivity expression. Obviously, such an exact solution may greatly prevent the frequency sensitivity from wide application in practical engineering, particularly, for large beam-like structures or complex sections of beams. Because of this limitation, this research is undertaken to provide an easy and direct approach for developing as well as calculating the design sensitivity for a general vibrating beam with the discrete method. Usually, when the cross-section of a beam is of non-uniformity, e.g., for a double-tapered beam, the solution has to be achieved by numerical approaches, most commonly by the FE method.

In the present work, a higher-order finite beam element model of two-node, three DOFs per node (3DPN) [4], namely, the transverse displacement, rotation (or slope) and curvature, is employed instead of the conventional finite beam element of two-node, two DOFs per node (2DPN) for the frequency sensitivity analysis. This is because the second derivative of the cubic polynomial displacement function of the 2DPN beam element, or the curvature, which is desirable in the present derivation with both inertias of a lumped mass under consideration, is intrinsically discontinuous at the inter-element node [4,13]. Consequently, this significant disadvantage of the shape functions adversely affects the correct derivation of the frequency sensitivity with use of the FE method suggested by the present author and coworkers [9]. With use of the 3DPN element, the discrete method on the FE basis can simply be used for development of the frequency sensitivity. Most importantly, within the scope of the same beam element model the sensitivity computation can be easily accomplished since the curvature is just one of the essential nodal DOFs of the 3DPN element and is then accessed directly. Additionally, the continuity of the curvature can be automatically guaranteed at an inter-element node with no attachment of a lumped mass.

In fact, the higher-order beam elements have long been utilized for vibration analysis of non-uniform beams [4,14,15]. It is a common knowledge that the higher-order beam element is superior to the lower-order element in the natural frequency extraction, especially for higher-order ones. In this work, the major attention is paid to the efficient evaluation of the frequency sensitivity of a beam-mass system except for its successful derivation. First, the FE method combined with the 3DPN model aforementioned is used for the sensitivity formulation. Based only on the characteristic features of the shape functions of quintic polynomials for the 3DPN element, which is higher enough for the present problem, a closed-form solution of the frequency sensitivity with respect to the mass-attached point is developed readily. Next, with use of the same element, the design sensitivity estimation is numerically executed for a uniform or linearly tapered beam carrying lumped masses. Numerical results show that the frequency sensitivity can be obtained rather precisely.

Although the 3DPN beam element is well suited to the sensitivity computation when the translational and rotary inertias of a lumped mass are both involved in the analysis, introducing an additional DOF of the curvature at a mesh node also makes the FE analysis quite a bit complicated in comparison with the classical 2DPN beam element. It is well known that the curvature of the transverse displacement function of a beam is discontinuous across the mass-attached point when the rotary inertial effect of the lumped mass is considered [1,2,5]. Therefore, two DOFs of the curvature have to be defined separately at a node with a mass attachment such that the assembly of the mass and stiffness matrices of the beam itself is highly related to the lumped mass position. For the purpose of efficient evaluation of the frequency sensitivity, further efforts are needed to find out an effective approach for using the 2DPN element, and a more precise calculation of the curvature should be delivered. Two alternative schemes are devised in this study to approximate the curvatures at a node with this element model. The applicability of the proposed methods is verified with slender beams of the uniform or linearly tapered cross-section, and the resultant accuracy of the frequency sensitivity is illustrated. Comparison of numerical solutions shows clearly that the sensitivity can, in general, be adequately estimated with use of the 2DPN beam element in a more refined FE discretization, and the results are generally more accurate only to the lower-order frequencies.

2. Frequency sensitivity formulation

In this section the FE method is briefly described for derivation of the frequency sensitivity of a beam-mass system with respect to the attachment position of a lumped mass with both the inertial effects included. It will be observed that this procedure is a natural development of the previous work [9] by applying formally the concept of shape function of a higher-order beam element.

2.1. Foundation of the FE method for the frequency sensitivity analysis

Consider a non-uniform beam of overall length L carrying a lumped (nonstructural) mass of the translational inertia M (magnitude) and the rotary inertia J (rotary moment of inertia about the neutral axis of the beam) at an intermediate arbitrary point $x=b$ measured from the left end of the beam, as shown in Fig. 1. Suppose that the beam-mass system is discretized with an adequate FE mesh. Then, from the equation of motion for the free undamped vibration of the beam, the characteristic equation of the natural frequency is represented by:

$$([K]-\omega_i^2\ [M])\{\phi\}_i=\{0\} \tag{1}$$

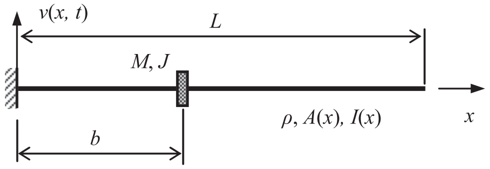

Fig. 1. A cantilever beam carrying a concentrated mass.