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Skew ray tracing and sensitivity analysis of hyperboloid optical boundary surfaces

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ABSTRACT

One of the most popular mathematical tools in fields of robotics, mechanisms and computer graphics is the 4×4 homogeneous transformation matrix. Our group's previous application of the homogeneous transformation matrix to flat and spherical optical boundaries has been extended to hyperboloid surfaces for: (1) skew ray tracing to determine the paths of reflected/refracted skew rays; (2) sensitivity analysis for direct mathematical expression of differential changes of incident points and reflected/refracted vectors with respect to changes in incident light sources and boundary geometric parameters; (3) a sensitivity analysis-based merit function derived directly from mathematical expression of catadioptric imaging system components. The presented methodology is highly suited to digital implementation and offers direct and rapid analytical statement of ray path, chief ray, marginal rays and merit functions of optical systems.

1. Introduction

Optical system design requires accurate determination of the paths of light rays interacting with reflective and refractive system components. This may be done by successive application of Snell's well-known laws of reflection and refraction, an extensively employed method known as ray tracing. The generalized version is known as skew ray tracing and is more difficult to perform, but analytical modeling and evaluation of optical systems is presently impossible without this technique. Traditional skew ray tracing, even with digital computers, is computationally very expensive. Our previous work expedited skew ray tracing by formulating it in terms of homogeneous transformation matrices and applied it to the easily-manufactured flat [1] and spherical [2] optical components commonly used by industry. However, aspherical boundary surfaces such as hyperboloid surfaces sometimes have significant advantages over spherical boundary surfaces. For example, it is known that parallel incident rays reflected by a spherical concave mirror undergo spherical aberration and do not converge at the focal point. However, a convex/concave hyperboloid reflecting mirror can converge these rays to form a virtual/real imaged point. Therefore, hyperboloids are used in a great variety of applications such as flashlights, automobile headlight reflectors, radiotelescope antennas, microwave horns, acoustical dishes and optical telescope mirrors.

Ray tracing though an aspherical surface is difficult since the intersection of a ray and an aspherical surface cannot be determined directly. Smith [3] performed aspherical-boundary skew ray tracing by a series of approximations which continued until approximation error became negligible, a computationally expensive procedure. This present work achieves more efficient aspherical-boundary skew ray tracing by use of Snell's laws formulated as the homogeneous transformation matrices shown in Section 2. Sensitivity analysis is presented in Section 3.

The discussion in Sections 2 and 3 is limited to monochromatic light. However, most light sources are polychromatic. When polychromatic light is refracted, each monochromatic component has its own unique interaction with the refractive components of the optical system. Each monochromatic component thus takes a different ray path through the system and each arrives at a slightly different position. The resulting image is different for different colors, an effect called chromatic aberration. Welford [4] pointed out that exact formulae for chromatic aberration are cumbersome, but Welford's numerical ray-tracing technique remains the universally adopted method for detailed analysis of optical systems. Even with computers and commercial software, the process is slow. Therefore, Section 4 provides algebraic

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expressions for chromatic aberration suitable for application in rapid computerized evaluation of optical system quality. Conclusions are presented in Section 5.

The 4×4 homogeneous transformation matrix is one of the most efficient and useful tools in robotics [5,6], mechanisms [7,8] and computer graphics [9]. In homogeneous coordinate representation, a position vector $P_{ix}\mathbf{i} + P_{iy}\mathbf{j} + P_{iz}\mathbf{k}$ is written as a column matrix ${}^j\mathbf{P}_i = [\,P_{ix} \quad P_{iy} \quad P_{iz} \quad 1\,]^T$. In the following, the pre-superscript "j" of the leading symbol ${}^j\mathbf{P}_i$ means this vector is referred with respect to coordinate frame $(xyz)_j$. Given a point ${}^j\mathbf{P}_i$, its transformation ${}^k\mathbf{P}_i$ is represented by the matrix product ${}^k\mathbf{P}_i = {}^k\mathbf{A}_j{}^j\mathbf{P}_i$, where ${}^k\mathbf{A}_j$ is a 4×4 matrix defining the position and orientation (referred to as pose matrix hereafter) of a frame $(xyz)_j$ with respect to another frame $(xyz)_k$ [5]. If the ith vector is referred with respect to the ith frame $(xyz)_i$ (i.e. ${}^i\mathbf{P}_i$ uses the same number as both its pre-superscript and post-subscript), then its pre-superscript "i" will be omitted for reasons of simplicity. These notational rules are also applied to unit directional vector ${}^j\ell_i = [\ell_{ix}\ \ell_{iy}\ \ell_{iz}\ 0]^T$.

2. Skew ray tracing at hyperboloid boundary surfaces

2.1. Generation of boundary surfaces and normals

One important feature of typical optical elements is that their boundary surfaces, at which the reflection and refraction processes occur, are surfaces of revolution. Consequently, the proposed methodology first studies the boundary surfaces in terms of revolution geometry and then establishes skew ray paths via the ray tracing technique. A boundary surface $\mathbf{r}_i$ can be obtained by rotating its generating curve $\mathbf{q}_i = \begin{bmatrix} x(\beta_i) & y(\beta_i) & 0 & 1 \end{bmatrix}^T$ about its y_i axis [10] (see Fig. 1) as

$$\mathbf{r}_i = \mathrm{Rot}(y_i, \alpha_i)\mathbf{S}_i = \begin{bmatrix} C\alpha_i & 0 & S\alpha_i & 0 \\ 0 & 1 & 0 & 0 \\ -S\alpha_i & 0 & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(\beta_i) \\ y(\beta_i) \\ 0 \\ 1 \end{bmatrix} = [\,x(\beta_i)C\alpha_i \quad y(\beta_i) \quad -x(\beta_i)S\alpha_i \quad 1\,]^T, \tag{1}$$

where $\mathrm{Rot}(y_i,\alpha_i)$ is the rotational transformation matrices about the respective y_i axis. The notations C and S denote cosine and sine, respectively. Thus, unit normal $\mathbf{n}_i$ along $\mathbf{r}_i$ is given by $\mathbf{n}_i = s_i((\partial^i\mathbf{r}_i/\partial\alpha_i) \times (\partial^i\mathbf{r}_i/\partial\beta_i))/|(\partial^i\mathbf{r}_i/\partial\alpha_i) \times (\partial^i\mathbf{r}_i/\partial\beta_i)|$ where s_i is set to +1 or −1, the choice being made in order to make the cosine of the incidence angle $C\theta_i > 0$. Consequently, unit normals along boundary surfaces $\mathbf{r}_i$ are given by

$$\mathbf{n}_i = s_i \begin{bmatrix} \dfrac{y'(\beta_i)C\alpha_i}{\sqrt{y'(\beta_i)^2 + x'(\beta_i)^2}} & \dfrac{-x'(\beta_i)}{\sqrt{y'(\beta_i)^2 + x'(\beta_i)^2}} & \dfrac{-y'(\beta_i)S\alpha_i}{\sqrt{y'(\beta_i)^2 + x'(\beta_i)^2}} & 0 \end{bmatrix}^T, \tag{2}$$

where $x'(\beta_i) = dx(\beta_i)/d\beta_i, y'(\beta_i) = dy(\beta_i)/d\beta_i$ and $z'(\beta_i) = dz(\beta_i)/d\beta_i$. Eqs. (1) and (2) give the parametric expression of boundary surface $\mathbf{r}_i$ of arbitrary revolution shape and its unit boundary normals $\mathbf{n}_i$ with respect to the ith optical frame $(xyz)i$. Once having these expressions, it is possible to trace any skew ray from reflective and refractive optical laws in terms of the boundary unit normal and the unit directional vector of an incident ray.

2.2. Reflection and refraction at boundary surfaces

Our group's previous application of the homogeneous transformation matrix to flat and spherical optical boundaries [1,2] required transformation of the calculated results all the way back to world coordinates to complete the ray tracing process. In order to simplify

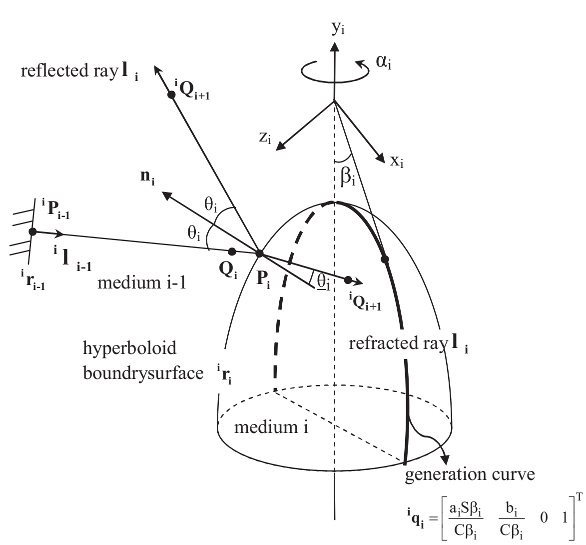

Fig. 1. A surface of revolution forms one boundary surface of a medium; schematic diagram of skew ray tracing at the boundary.