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Global sensitivity analysis for dynamic systems with stochastic input processes

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ABSTRACT

Dynamic systems with stochastic input processes are general engineering models. In order to apply the global sensitivity analysis on these systems to aid safety design, variance based sensitivity indices are extended to stochastic processes in this paper. Factor sensitivity indices and time sensitivity indices are defined to describe the contributions of the input factor variances of the whole time and a specific time to the output variance at the current time, respectively. Analytical expressions of sensitivity indices for linear time-invariant systems with Gaussian processes are derived due to the extensive use of them, including continuous-time systems and discrete-time systems. Monte Carlo simulation is used to verify the derived analytical expressions. The derived expressions could be used directly in practical applications to reduce computation cost. The applications of the sensitivity indices have been discussed with engineering examples, which include transient temperature measurements and a forced vibration system. Some safety design measures aided by the sensitivity indices are also discussed in these two examples.

1. Introduction

Sensitivity analysis is an essential part of engineering system modeling [1] and can be used in design verification to make sure that the product would operate safely [2–4]. System factors with epistemic uncertainty from a lack of knowledge or aleatory uncertainty from an inherent randomness would result in the uncertainty of system outputs [5–9]. Sensitivity analysis aims at ranking the system factors qualitatively or quantitatively with respect to their contributions to the system output uncertainty. Then more effects could be focused on these most important factors to improve the design safety level. Supplementary information should be provided to reduce the uncertainty of these factors or even design should be changed to accommodate the uncertainty [2].

Probabilistic method is the common tool used for representing the uncertainties though alternative uncertainty representations (e.g., evidence theory, possibility theory, interval analysis) are active areas of research [10]. In a probabilistic framework, both epistemic uncertainty and aleatory uncertainty could be modeled as random variables or stochastic processes [11–13]. The sensitivity analysis approaches may be local or global. Local approaches are applicable when the variations around the midpoints of the factors are small while global approaches allow all the factors vary simultaneously over their entire ranges. When the model is non-linear and various input variables are affected by uncertainties of different orders of magnitude, global methods should be used [1]. Global approaches have been further divided into three categories [14–16], i.e., non-parameter techniques, variance based methods, and moment-independent indicators. The variance based methods have been extensively used in many engineering cases [1,17–19].

For static models, if the system factors are independent, the model function could be split into a sum of functions of increasing dimension, known as high dimensional model representation (HDMR) [20]. With this decomposition, sensitivity indices are calculated. For systems with dependent and correlated factors, many methods were proposed, including alternative decomposition [21], introducing new indices [22] and generalization of the variance based sensitivity indices [23].

Recently, variance-based sensitivity analysis has also been applied to dynamic models since most physical systems could be described by differential equations. In the dynamic systems, if the probabilistic distributions of factors are not related to time, the factors could still be modeled as random variables and, at each time instant, the output of the model could still be decomposed into summands of increasing dimension. In this way, Ref. [24] presents the global sensitivity analysis for dynamic models based

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on HDMR and polynomial chaos expansion of the output. In addition, Ref. [25] provides a method employing principal components analysisis combined with analysis of variance. These methods focused on random model parameters modeled by random variables. However, in dynamic systems, many systems factors are often related with time and should be modeled as stochastic processes due to either epistemic uncertainty (e.g., Brownian motion [26]) or aleatory uncertainty (e.g., subsurface reservoir [12]). Hence, sensitivity analysis methods for stochastic processes are necessary. For example, Ref. [27] summarized three analyses and performed sensitivity analyses by using partial rank correlation coefficients, stepwise rank regression analyses, and scatterplots. Moreover, for a dynamic model described by differential equations or difference equations, the outputs at a time instant could be affected not only by the inputs at the same time instant but also by the inputs before that instant. This feature of dynamic models requires that the sensitivity analysis should quantify not only the importance of different factors but also the importance of the same factor at different time instants. However, few variance based sensitivity analysis approaches have been proposed in the literature to deal with these aspects of stochastic processes.

This paper extends the variance based sensitivity indices to stochastic processes and defines factor sensitivity indices and time sensitivity indices. Factor sensitivity indices describes the contributions of the whole variances for each input to the current output variance, while time sensitivity indices represent the contributions of the variance for a specific input factor at a certain time instant to the current output variance. Considering that linear time-invariant (LTI) system is a kind of basic system model in many engineering fields (e.g., mechanical vibration [28], electrical communication [29], automation control [30] and heat transfer [31]) and it is used frequently in practical engineering modeling, sensitivity indices for this kind of system are further studied. Specifically, analytical expressions sensitivity indices are derived for LTI systems with Gaussian processes for the convenience of practical application.

The outline of this paper is as follows. Variance based sensitivity indices for stochastic processes in dynamic systems are defined in Section 2 and sensitivity indices for LTI system with Gaussian processes are derived analytically in Section 3, including continuous-time systems and discrete-time systems. Section 4 verifies the analytical expressions by numerical cases. Section 5 applies the expressions to engineering examples. Some conclusions are drawn in Section 6.

2. Global sensitivity analysis

Consider a dynamic system with multiple outputs $\boldsymbol{Y}(t)=[Y_1(t), Y_2(t),\ldots, Y_{nY}(t)]^T$ (the superscript T denotes matrix transposition) described by the following vector differential equation with a random inhomogeneous part

$$\dot{\boldsymbol{Y}}(t)=\boldsymbol{f}(t)\boldsymbol{Y}(t)+\mathbf{g}(t,\boldsymbol{X}(t)); \quad \boldsymbol{Y}(t_0)=\boldsymbol{Y}_0, \tag{1}$$

where $\boldsymbol{f}(t)$ is an $nY\times nY$ real matrix whose elements are functions of t, and $\mathbf{g}(\bullet)$ is a vector function of t and stochastic input factors $\boldsymbol{X}(t)=[X_1(t), X_2(t),\ldots, X_{nX}(t)]$. The input factors $[X_1(t), X_2(t), \ldots, X_{nX}(t)]$ are considered as independent, real and continuous-state stochastic processes. Indeed, Eq. (1) describes linear systems for $\boldsymbol{Y}(t)$ subject to random inputs [32]. It is worth noting that Eq. (1) is applicable for continuous systems, while for discrete systems the corresponding difference equation should be discussed in similar ways and in Section 3 results for discrete LTI systems would be provided. As a convention, when referring to random variables or stochastic processes, we use upper-case letters (e.g., X, Y) and lower-case letters (e.g., x, y) to represent their generic aspects and their observed values, respectively. According to the theory of stochastic process [26], the statistical properties of each $X_i(t)$, $i=1,2,\ldots,nX$, are completely determined in terms of its nth-order distribution (There are processes for which this is not true. However, such processes are mainly of mathematical interest [26])

$$F(x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)};t_1,t_2,\ldots,t_n)=P\{x_i(t_1)\le x_i^{(1)},x_i(t_2)\le x_i^{(2)},\ldots,x_i(t_n)\le x_i^{(n)}\}, \tag{2}$$

and the corresponding probability density function (p.d.f.) equals

$$p_i(x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)};t_1,t_2,\ldots,t_n)=\frac{\partial^n F(x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)};t_1,t_2,\ldots,t_n)}{\partial x_i^{(1)}\partial x_i^{(2)}\ldots\partial x_i^{(n)}} \tag{3}$$

Particularly, $p_i(x_i, t)$ represents the p.d.f. of $X_i(t)$ at time t.

According to the theory of stochastic differential equations [32], the unique and mean square solution of Eq. (1) can be represented by

$$\boldsymbol{Y}(t)=\boldsymbol{b}(t,t_0)\boldsymbol{Y}_0+\int_{t_0}^{t}\boldsymbol{b}(t,s)\mathbf{g}(s,\boldsymbol{X}(s))ds, \tag{4}$$

where $\boldsymbol{b}(t,t_0)$ is the principal matrix associated with the corresponding homogeneous equation.

For the sake of simplicity, here we focus on one output, denoted by $Y(t)$. $Y(t)$ could be any one of $\boldsymbol{Y}(t)$ and the others could be discussed in similar ways. Based on Eq. (4), $Y(t)$ could be expressed as

$$Y(t)=\boldsymbol{b}_{1\times nY}(t,t_0)\boldsymbol{Y}_0+\int_{t_0}^{t}\boldsymbol{b}_{1\times nY}(t,s)\mathbf{g}(s,\boldsymbol{X}(s))ds, \tag{5}$$

where $\boldsymbol{b}_{1\times nY}(t,t_0)$ refers to the corresponding row of $\boldsymbol{b}(t,t_0)$ according to the position of $Y(t)$ in $\boldsymbol{Y}(t)$. Hence, the output $Y(t)$ at a specific time t_Y, denoted by $Y(t_Y)$, could expressed as

$$Y(t_Y)=T(t_Y;t,\boldsymbol{X}(t))=T[t_Y;t,X_1(t),X_2(t),\ldots,X_{nX}(t)], \tag{6}$$

where $T(\bullet)$ denotes an operator of t and $\boldsymbol{X}(t)$ with parameter t_Y. It is worth noting that the output $Y(t_Y)$ at time t_Y is affected by the inputs before t_Y, i.e., the whole processes of $\boldsymbol{X}(t)$ ($t_0\le t\le t_Y$).

Similar to the function decomposition proposed by Sobol' [20] and developed by Saltelli [1], the operator $T(\bullet)$ in (6) could also be decomposed into summands of increasing dimensionality, namely

$$\begin{aligned}Y(t_Y)&=T(t_Y;t,\boldsymbol{X}(t))\\&=T_0(t_Y;t)+\sum_{i=1}^{nX}T_i(t_Y;t,X_i(t))+\sum_{1\le i<j\le nX}T_{ij}(t_Y;t,X_i(t),X_j(t))\\&\quad+\cdots+T_{12\cdots nX}(t_Y;t,X_1(t),X_2(t),\ldots,X_{nX}(t)).\end{aligned} \tag{7}$$

The summands of Eq. (7) are given by

$$\begin{aligned}T_0(t_Y;t)&=\int T(t_Y;t,\boldsymbol{X}(t))p(\boldsymbol{x},\ t)d\boldsymbol{x},\\T_i(t_Y;t,X_i(t))&=\int T(t_Y;t,\boldsymbol{X}(t))p_{-i}(\boldsymbol{x}_{-i},t)d\boldsymbol{x}_{-i}-T_0(t_Y;t),\\T_{ij}(t_Y;t,X_i(t),X_j(t))&=\int T(t_Y;t,\boldsymbol{X}(t))p_{-ij}(\boldsymbol{x}_{-ij},t)d\boldsymbol{x}_{-ij}-T_i(t_Y;t,X_i(t))\\&\quad-T_j(t_Y;t,X_j(t))-T_0(t_Y;t),\ldots,\end{aligned} \tag{8}$$

where $\boldsymbol{x}_{-i}$ and $\boldsymbol{x}_{-ij}$ denote all variables except x_i, and x_i, x_j, respectively. Here, $p(\boldsymbol{x},t)$ is the joint p.d.f. for $\boldsymbol{x}$ at time t and satisfies

$$p(\boldsymbol{x},\ t)=\prod_{i=1}^{nX}p_i(x_i,t), \tag{9}$$

when the stochastic processes are independent. Similarly, $p_{-i}(\boldsymbol{x}_{-i}, t)$ and $p_{-ij}(\boldsymbol{x}_{-ij}, t)$ are marginal p.d.f.s obtained by integrating the