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The vanishing viscosity method for the sensitivity analysis of an optimal control problem of conservation laws in the presence of shocks

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ABSTRACT

In this article we study, by the vanishing viscosity method, the sensitivity analysis of an optimal control problem for 1-D scalar conservation laws in the presence of shocks. It is reduced to investigate the vanishing viscosity limit for the nonlinear conservation law, the corresponding linearized equation and its adjoint equation, respectively. We employ the method of matched asymptotic expansions to construct approximate solutions to those equations. It is then proved that the approximate solutions, respectively, satisfy those viscous equations in the asymptotic sense, and converge to the solutions of the corresponding inviscid problems with certain convergent rates. A new equation for the variation of shock positions is derived. It is also discussed how to identify descent directions to find the minimizer of the viscous optimal control problem in the quasi-shock case.

1. Introduction

Optimal control for hyperbolic conservation laws requires a considerable analytical effort and computational expense in practice, is thus a difficult topic. Some methods have been developed in the last several years to reduce the computational cost and to render this type of problems affordable. In particular, recently in [1] Castro, Palacios and Zuazua have developed an "alternating descent method" that takes the possible shock discontinuities into account, for the optimal control of the inviscid Burgers equation in one space dimension. Further in [2] this numerical method is also employed to study the optimal control problem of the Burgers equation with small viscosity via the Hopf–Cole formula which can be found in [3,4], for instance.

In the present article, we study the sensitivity analysis of an optimal control problem for 1-D general nonlinear scalar conservation law in the presence of shocks by the method of vanishing viscosity. It reduces to study the vanishing viscosity limit of solutions to the nonlinear conservation law, the corresponding linearized equation and its adjoint equation, which is the main result of this paper (see Theorem 4.1). Observing that the discontinuities in those equations will lead to difficulties when passing to the limit, we apply the method of matched asymptotic expansions to study the convergence. In particular, the solutions to the viscous adjoint problem approach a constant in a "triangular region" formed by the characteristics

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intersecting at the shock, as the viscosity vanishes. Thus it extends the results in [2] for the Burgers equation to a general case. Moreover, certain convergent rates are obtained so that it generalizes the result of James and Sepúlveda (see [5]) for adjoint equations. As a result, it is reasonable to apply the efficient numerical method, such as the "alternating descent method" developed in [2], to study the optimal control problem for general conservation laws with small viscosity, when the solutions to the corresponding hyperbolic conservation laws have shock discontinuities. Another main feature of this article is that, we have derived a new equation for the variation of shock position, which approaches, as a parameter tend to infinity, to the one obtained by Bressan and Marson (see [6]).

We first state the optimal control problem as follows. For a given $T > 0$, we study the following inviscid problem

$$u_t + \big(F(u)\big)_x = 0, \quad \text{in } \mathbb{R} \times (0, T), \tag{1.1}$$

$$u(x, 0) = u^I(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a smooth function, $f = F_u$ and u^I is a piecewise smooth function. A function $u(x, t)$ is called a *single-shock* solution (see e.g., [7]) to (1.1)–(1.2) up to time T if

1. $u(x, t)$ is a distributional solution of the hyperbolic system (1.1)–(1.2) in the region $\mathbb{R} \times [0, T]$.
2. There exists a smooth curve, the shock, $x = \varphi(t),\ 0 \le t \le T$, so that $u(x, t)$ is sufficiently smooth at any point $x \ne \varphi(t)$.
3. The limits

$$\partial_x^k u(\varphi(t) - 0, t) = \lim_{x\to\varphi(t)-0} \partial_x^k u(x, t), \qquad \partial_x^k u(\varphi(t) + 0, t) = \lim_{x\to\varphi(t)+0} \partial_x^k u(x, t),$$

exist and are finite for $t \le T$ and $k = 0, 1, 2, 3, 4$.

4. The entropy condition is satisfied at $x = \varphi(t)$, that is,

$$f(u(\varphi(t) - 0, t)) > \varphi'(t) > f(u(\varphi(t) + 0, t)).$$

Although we present the results for single-shock solutions, it will be clear from our analysis that similar results hold for piece-wise smooth solutions with finitely many non-interacting shocks.

Given a target function $u^D \in L^2(\mathbb{R})$ we consider the cost functional to be minimized $J : L^1(\mathbb{R}) \to \mathbb{R}$, defined by

$$J(u^I) = \int_{\mathbb{R}} \left|u(x, T) - u^D(x)\right|^2 dx, \tag{1.3}$$

where $u(x, t)$ is the unique *single-shock* solution to (1.1)–(1.2).

We also introduce the set of admissible initial data $\mathcal{U}_{ad} \subset L^1(\mathbb{R})$, that we shall define later in order to guarantee the existence of the following optimization problem:

Find $u^{I,\min} \in \mathcal{U}_{ad}$ such that

$$J(u^{I,\min}) = \min_{u^I\in\mathcal{U}_{ad}} J(u^I).$$

Such a problem has been studied in e.g. [1,2] in the case that $F(u) = \frac{u^2}{2}$. This is one of the model optimization problems that is often addressed in the context of optimal aerodynamic design (see [8]).

In practical applications, in order to perform numerical computations and simulations one has to replace the continuous optimization problem above by a discrete approximation, and develop efficient algorithms for computing accurate approximations of the discrete minimizers. The most efficient methods to approximate minimizers are the gradient methods: one first linearizes (1.1) to obtain a descent direction of the continuous functional J, then takes a numerical approximation of this descent direction with the discrete values provided by the numerical scheme. To this end, we first linearize (1.1) to yield

$$\partial_t(\delta u) + \partial_x(f(u)\delta u) = 0, \quad \text{in } \mathbb{R} \times (0, T), \tag{1.4}$$

where δu is the variation of u with respect to u^I. However, it is not justified when the solutions have shocks since singular terms may appear on the linearization over the shock location. Accordingly in optimal control applications we also need to take into account the sensitivity for the shock location (which has been studied by many authors, see, e.g. [1,2,6,9,10]). Roughly speaking, the main conclusion of that analysis is that the classical linearized equation (1.4) must be complemented with an equation for the sensitivity of the shock positions (see Section 3.2).

Moreover, it is also necessary to carry out the sensitivity analysis of the optimal control problem for the numerical purpose. As we shall see in Section 2.3, it reduces to study the following (inverse) adjoint problem corresponding to (1.4),

$$-p_t - f(u)p_x = 0, \quad \text{in } \mathbb{R} \times (0, T), \tag{1.5}$$

$$p(x, T) = p^T(x) \equiv u(x, T) - u^D(x), \quad x \in \mathbb{R}. \tag{1.6}$$

Next, we introduce accordingly the viscous problem corresponding to (1.1) as

$$u_t^{\nu,\varepsilon} + (F(u^{\nu,\varepsilon}))_x = \nu u_{xx}^{\nu,\varepsilon}, \quad \text{in } \mathbb{R} \times (0, T), \tag{1.7}$$

$$u^{\nu,\varepsilon}|_{t=0} = g^{\nu,\varepsilon}, \tag{1.8}$$