Composites: Part B 56 (2014) 840–846

Contents lists available at ScienceDirect

Composites: Part B

journal homepage: www.elsevier.com/locate/compositesb

Buckling and sensitivity analysis of nonlocal orthotropic nanoplates with uncertain material properties

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ARTICLE INFO

Article history:
Received 20 March 2013
Accepted 12 August 2013
Available online 22 August 2013

Keywords:
A. Nano-structures
A. Plates
B. Buckling
C. Analytical modeling
Material uncertainty

ABSTRACT

Accurate estimates of the orthotropic properties of nano-materials are usually not available due to the difficulties in making measurements at nano-scale. However the values of the elastic constants may be known with some level uncertainty. In the present study an ellipsoidal convex model is employed to study the biaxial buckling of a rectangular orthotropic nanoplate with the material properties displaying uncertain-but-bounded variations around their nominal values. Such uncertainties are not uncommon in nano-sized structures and the convex analysis enables to determine the lowest buckling loads for a given level of material uncertainty. The nanoplate considered in the present study is modeled as a nonlocal plate to take the small-size effects into account with the small-scale parameter also taken to be uncertain. Method of Lagrange multipliers is applied to obtain the worst-case variations of the orthotropic constants with respect to the critical buckling load. The sensitivity of the buckling load to the uncertainties in the elastic constants is also investigated. Numerical results are given to study the effect of material uncertainty on the buckling load.

1. Introduction

In the deterministic analysis, variations in the material properties are neglected and the average values of the elastic constants are used to obtain a mean value for the structural response. This approach does not take the deviations from the average into account even though it is usually difficult to determine the properties of a material with any certainty. This is more so for nano-sized structures which exhibit large variations in their material properties due to defects and imperfections in their molecular structures. Moreover experimental difficulties in making accurate measurements at the nano-scale lead to significant scatter in the values of elastic constants. For example the values of Young's modulus of carbon nanotubes have been reported between 1 and 5 TPa in the literature [1–3].

Nanoplates, made of mono or multilayer graphene, are often employed in nanotechnology applications as sensors [4] and actuators [5] as well as in many other capacities and their usage is expected to increase [6]. Quite often they are subject to in-plane loads making them susceptible to buckling due to their extremely small thickness measured in nanometers. This situation has led to several studies on the subject and the buckling of single layer graphene has been studied in [7] without taking small-scale effects into account and in [8,9] employing the nonlocal theory. Buckling of isotropic nanoplates has been studied in [10,11] and orthotropic plates in [12–17] employing nonlocal constitutive relations and taking various effects such as nonuniform thickness [10], temperature [13], shear deformation [14] and nonuniform in-plane loads [15] into account. Variational principles for vibrating multi-layered orthotropic graphenes sheets were given in [16,17]. Studies on the vibrations of orthotropic nanoplates include [18–21]. The nonlocal theory developed in the 1970s [22,23] includes the small-scale effects by expressing stress as a function of strain at all points of the continuum.

Buckling and vibration results given in [10–21] for graphene and nanoplates are based on the deterministic values of the elastic constants and as such neglect the variations in the material properties even though such variations are common. Nominal buckling load, corresponding to a deterministic model, could be higher than the applied compressive loads, indicating a safe design. However, a safe design based on deterministic material values is not robust due to inherent uncertainties in elastic constants [24]. This situation necessitates taking the data uncertainties into account in a non-deterministic model which will improve the reliability of the results by providing conservative load-carrying estimates.

Such a model could be probabilistic or statistical requiring information on the probability distributions of random variables.

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http://dx.doi.org/10.1016/j.compositesb.2013.08.054

Obtaining this information in many cases is a difficult task. However, data on upper and lower bounds of uncertain parameters may be known or can be estimated with reasonable accuracy in which case an approach based on convex modeling would yield the reliable results. In this case the total level of uncertainties is bounded by an n-dimensional ellipsoid where the number of dimensions is equal to the number of uncertain parameters [25]. Examples of convex modeling applied to engineering problems with uncertain data include [24–32]. A comparison of convex modeling with probabilistic methods is given in [33] and the book by Ben-Haim and Elishakoff [34] details the techniques of convex modeling.

The present study involves the computation of the buckling load of an orthotropic nanoplate in the presence of material uncertainties using convex modeling with the L_2 norm of the uncertainties bounded. The constitutive relations are based on the nonlocal theory of plates which takes nano-scale effects into account. The sensitivity of the critical load to uncertainty is also investigated by defining relative sensitivities in terms of uncertainty parameters [35,36]. Further information on sensitivity indices can be found in [37–39]. Numerical results are given to investigate the effect of uncertainty on the buckling loads and the dependence of the relative sensitivities on the aspect ratio is studied by means of contour plots.

2. Convex modeling

In this section, the method of solution to compute the uncertainty parameters corresponding to the least-favorable buckling load is summarized. Let $\widetilde{U}_i = U_{0i}(1+\gamma_i)$ denote the ith uncertain material property ($i = 1,2,\ldots,n$) where U_{0i} is the nominal value of $\widetilde{U}_i$ and γ_i is an unknown parameter to be computed to minimize (the least favorable solution) or to maximize (the most favorable solution) the critical buckling load. The convex model of uncertainties can be described by an n-dimensional uncertainty vector $\Phi = (\gamma_1,\gamma_2,\gamma_3,\ldots,\gamma_n)^T$ defined on a convex set S such that $\Phi \in S$. In the present study uncertainty parameters γ_i belong to a bounded quadratic convex set defined as

$$S(\Phi,\beta) = \{\Phi | \Phi \in R^n, \Phi^T\Phi \leqslant \beta^2\} \tag{1}$$

where β is the prescribed measure of uncertainty satisfying the inequality $\beta < 1$. Thus

$$S(\Phi,\beta) = \left\{\Phi | \Phi \in R^n, \sum_i^n \gamma_i^2 \leqslant \beta^2\right\} \tag{2}$$

Here β is the radius of the n-dimensional ellipsoid. The problem investigated involves the computation of the unknown parameters γ_i such that the buckling load becomes the lowest or highest possible subject to the constraint $\sum_i^n \gamma_i^2 \leqslant \beta^2$. The buckling load can be expressed as a function of the uncertain quantities $\widetilde{U}_i$, *viz.*

$$N_{cr} = f(\widetilde{U}_i) \tag{3}$$

Since the variations of the uncertain parameters around their nominal values are small, the function $f(\widetilde{U}_i)$ can be expanded around U_{0i} by substituting $\widetilde{U}_i \cong U_{0i}(1+\gamma_i)$ into Eq. (3) and keeping only the terms which are of zero and first order in γ_i and neglecting the higher order terms. This computation can be carried out using the Taylor series expansion of the expression

$$(1 \pm \varepsilon)^c \cong (1 \mp c\varepsilon) + O(\varepsilon^2) \tag{4}$$

where the superscript c can take positive or negative values and $|\varepsilon| \ll 1$. The expansion of the buckling load around U_{0i} using Eq. (4) leads to

$$N_{cr} \cong f_0(U_{01},U_{02},\ldots,U_{0n}) + \sum_{i=1}^n f_i(U_{01},U_{02},\ldots,U_{0n})\gamma_i \tag{5}$$

where $f_0(U_{01},U_{02},\ldots,U_{0n})$ is the deterministic (nominal) value of the buckling load. The solutions are obtained by solving the following optimization problems

$$\min_{\gamma_i}\sum_{i=1}^n f_i(U_{01},U_{02},\ldots,U_{0n})\gamma_i \text{ and } \max_{\gamma_i}\sum_{i=1}^n f_i(U_{01},U_{02},\ldots,U_{0n})\gamma_i \tag{6}$$

subject to the constraint $\sum_i^n \gamma_i^2 \leqslant \beta^2$. It is noted that every affine functional whose domain is a compact convex set takes on its maximum value on the set of extreme points of its domain which is the boundary of the ellipsoid in the present case [31,32]. Thus the inequality (2) can be replaced by the equality

$$\sum_{i=1}^n \gamma_i^2 = \beta^2 \tag{7}$$

and the problem can be solved by introducing the Lagrangian

$$L(\widetilde{U}_i,\gamma_i) = \sum_{i=1}^n f_i(U_{01},U_{02},\ldots,U_{0n})\gamma_i + \lambda\left(\sum_{i=1}^n \gamma_i^2 - \beta^2\right) \tag{8}$$

where λ is the Lagrange multiplier. By taking the derivatives of $L(\widetilde{U}_i,\gamma_i)$ with respect to γ_i, we obtain

$$\frac{\partial L(\widetilde{U}_i,\gamma_i)}{\partial \gamma_i} = f_i(U_{01},U_{02},\ldots,U_{0n}) + 2\lambda\gamma_i = 0 \tag{9}$$

Thus

$$\gamma_i = -\frac{1}{2\lambda} f_i(U_{01},U_{02},\ldots,U_{0n}), \quad i = 1,2,\ldots n \tag{10}$$

The Lagrange multiplier is computed by substituting Eq. (10) into Eq. (7). This computation gives

$$\lambda = \pm\frac{1}{2\beta}\sqrt{\sum_{i=1}^n (f_i(U_{01},U_{02},\ldots,U_{0n}))^2} \tag{11}$$

In Eq. (11) the plus and minus values of λ correspond to the minimum and maximum values of the buckling load. The uncertain parameters γ_i are computed from Eqs. (10) and (11).

3. Orthotropic nanoplate with material uncertainty

The basic formulation is given next for an orthotropic rectangular plate subject to material uncertainty and biaxial buckling loads. Let $\widetilde{E}_1$ and $\widetilde{E}_2$ denote Young's moduli in the material coordinates, $\widetilde{G}_{12}$ in-plane shear modulus, and $\tilde{\nu}_{12}$ and $\tilde{\nu}_{21}$ in-plane Poisson's ratios where a tilde indicates an uncertain quantity. The plate has the length a, width b and thickness h in the x, y and z directions, respectively, with the aspect ratio denoted as $r = a/b$. The axial loads in the x and y directions are denoted as N_x and N_y, and the deflection function as $w(x,y;\widetilde{\Psi})$ where $\widetilde{\Psi}$ is the set of uncertain material constants defined as

$$\widetilde{\Psi} = \{\widetilde{\Psi} | \widetilde{E}_1, \widetilde{E}_2, \widetilde{G}_{12}, \tilde{\nu}_{12}, \tilde{\eta}\} \tag{12}$$

noting $\tilde{\nu}_{21} = \tilde{\nu}_{12}\widetilde{E}_2/\widetilde{E}_1$. The differential equation governing the biaxial buckling of an orthotropic nanoplate based on nonlocal constitutive relations can be expressed as [12]

$$\widetilde{D}_{11}\frac{\partial^4 w}{\partial x^4} + 2(\widetilde{D}_{12} + 2\widetilde{D}_{66})\frac{\partial^4 w}{\partial x^2\partial y^2} + \widetilde{D}_{22}\frac{\partial^4 w}{\partial y^4} + (1 - \tilde{\eta}^2\nabla^2)\left(N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{13}$$

where

$$\widetilde{D}_{11} = \frac{d\widetilde{E}_1}{1 - \tilde{\nu}_{12}\tilde{\nu}_{21}}, \quad \widetilde{D}_{12} = \frac{d\tilde{\nu}_{12}\widetilde{E}_2}{1 - \tilde{\nu}_{12}\tilde{\nu}_{21}}, \quad \widetilde{D}_{22} = \frac{d\widetilde{E}_2}{1 - \tilde{\nu}_{12}\tilde{\nu}_{21}}, \quad \widetilde{D}_{66} = d\widetilde{G}_{12}, \quad d = \frac{h^3}{12} \tag{14}$$