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Sensitivity analysis based crack propagation criterion for compressible and (near) incompressible hyperelastic materials

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ABSTRACT

Sensitivity analysis of an XFEM crack propagation model is developed for shape and material parameters, where the direct differentiation method is applied to large strain problems with hyperelastic neo-Hookean materials. The presence of level set functions to describe the crack position requires the development of a proper differentiation technique which is also addressed. In order to compute the analytical derivatives of such a complex numerical model the capabilities of the symbolic system AceGen are employed.

A crack propagation criterion based on the sensitivity formulation is developed, allowing the direct calculation of the crack growth length and direction without post-processing. Special attention is paid to the ability of satisfying incompressibility and near-incompressibility conditions.

The performance of the XFEM sensitivity analysis is assessed by the Cook's Membrane and Pre-crack Plate benchmark tests where sensitivities of displacements and crack propagation criteria based on potential energy have been analysed with respect to crack length and crack growth parameters. The techniques presented in this paper can be extended to anisotropic materials and non-linear materials exhibiting plasticity and viscoplasticity. Additionally, this formulation constitutes a base for further analysis of crack branching and crack joining problems.

1. Introduction

Numerical simulations have been gaining importance in industry as a valuable decision support tool in the design of engineering materials, components and systems, while saving resources. Nevertheless, to develop accurate and feasible numerical models it is essential to understand the influence of the different parameters on the overall response of a model. Through the sensitivity analyses it is possible to compute a rate of performance change with respect to the model design parameters. In general, the derivatives of an arbitrary response functional are calculated with respect to chosen parameters, such as model inputs (material constants, shape parameters, etc.) or intermediate or final results of the analysis (solution vectors, stress tensor, damage factors, etc.) [19,20,7]. Among the different methods available for sensitivity analysis, the direct differentiation technique, intensively developed in the work of Michaleris [25], is considered in this paper.

Sensitivity analyses are often applied in the context of shape optimisation problems, in which the behaviour of an energy functional is analysed when the shape of a body is modified [14,15]. This approach may be equally applied to fracture problems, as crack growth may be interpreted as a change of the shape of the body. First works in this direction have been developed by Hellen [16] and Parks [33] which evaluate the energy release change in a finite element problem when a crack is extended by a small amount. Nevertheless these works were based on specific examples, with an analytical or experimental solution available, and not on a fully developed sensitivity analysis formulation. Later, Zumwalt and El-Sayed [11] incorporated sensitivity analysis by calculating directly the derivative of the finite element stiffness matrix, however their work was restricted to specific finite element types. Another interesting approach was developed by Feijóo et al. [12] in which sensitivity analysis is used to obtain the expression for the energy release rate in a three-dimensional cracked body. As a drawback, this model requires the construction of an approximation for the velocity vector field, depending on some additional parameters.

This work is related to the described approaches, as one of the main objectives is to build a robust crack criterion based on sensitivity analysis. Nevertheless, it differs from the existing works, as the sensitivity of the potential energy with respect to the crack length is obtained through direct differentiation with

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exact derivatives. Moreover, the formulation is not restricted to a particular element type, is able to handle non-linear materials and large deformations and pays special attention to the satisfaction of incompressibility and near-incompressibility conditions. The cracks are described through the XFEM and the crack path is tracked by level set functions [31,30]. The presence of these kind of functions demands careful handling, as the derivatives required for the sensitivity calculations are not correctly evaluated by the simple application of the chain rule.

The XFEM [1,26] is one of the most prominent numerical techniques used in the field of fracture, which avoids the computationally expensive task of remeshing, each time a crack propagates. A large amount of literature concerning the XFEM is available in these days from applications to linear elastic fracture [1,26,9,43], continuum–discontinuum transition [39,2,5,3] and ductile fracture [35,36] to name a few. Nevertheless, in general, the sensitivity of different parameters used in the various XFEM formulations is only qualitatively evaluated. Problems with an available analytical solution are often used to determine the variations of the response of a certain numerical model under particular conditions, as the derivation of general sensitivity terms may become quite complex in the presence of non-linearities. Here tools are provided for efficient quantitative evaluation of the sensitivities associated with crack propagation problems modelled with the XFEM, taking advantage of the modern symbolic and algebraic computer systems. In particular, the AceGen [23] system allows the treatment of the equations associated with a crack propagation problem at a high abstract level. Furthermore, it includes an advance automatic differentiation tool, with simultaneous stochastic simplification of numerical code [45,24,22,21], allowing the generation of highly efficient FEM codes for analysis of both the primal problem and the subsequent sensitivity analysis.

This paper is organised as follows. In Section 2 the general sensitivity problem is described. The specific crack propagation problem is formulated in Section 3, focussing on the kinematic quantities and crack description through the XFEM. The crack propagation criterion is developed in Section 4. In Section 5 some numerical implementation details are given. Finally in Section 6 some numerical examples are presented and in Section 7 the main conclusions are outlined.

2. Sensitivity analysis through direct differentiation

The objective of sensitivity analysis is to evaluate the influence of a certain design parameter in the global response of a finite element problem.

In particular, non-linear finite element problems are often solved according to a Newton–Raphson iterative scheme [17,18,8]. A certain problem may be expressed in a residual form as follows:

$$\boldsymbol{R}(\boldsymbol{p}) = \boldsymbol{0} \tag{1}$$

where $\boldsymbol{R}$ is the residual and $\boldsymbol{p}$ is the vector of unknowns. As in general, in the current iterative solution, $\boldsymbol{p}^i$, $\boldsymbol{R}(\boldsymbol{p}^i) \neq \boldsymbol{0}$, the residual is updated according to

$$\boldsymbol{R}(\boldsymbol{p}^{i+1}) \approx \boldsymbol{R}(\boldsymbol{p}^i) + \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}\boldsymbol{p}}(\boldsymbol{p}^i)\partial\boldsymbol{p} = \boldsymbol{0} \tag{2}$$

where $\mathrm{d}\boldsymbol{R}/\mathrm{d}\boldsymbol{p}$ is the tangent stiffness matrix and $\partial\boldsymbol{p}$ is the incremental response, which may be determined as follows:

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}\boldsymbol{p}}(\boldsymbol{p}^i)\partial\boldsymbol{p} = -\boldsymbol{R}(\boldsymbol{p}^i) \tag{3}$$

The response of the system depends on a set of design parameters, $\boldsymbol{\psi}$, such as material constants, shape parameters or damage factors,

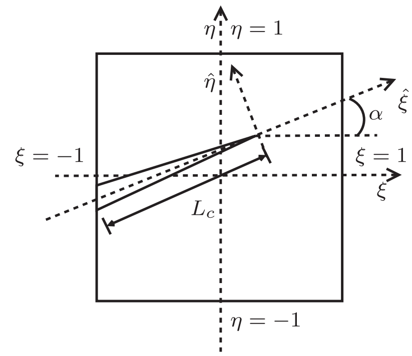

Fig. 1. Crack tip coordinates.

therefore, the residual may be re-written as a function of these parameters as follows:

$$\boldsymbol{R}(\boldsymbol{p}(\boldsymbol{\psi}), \boldsymbol{\psi}) = \boldsymbol{0} \tag{4}$$

In addition, a general response functional, $\boldsymbol{F}$, may be defined as

$$\boldsymbol{F}(\psi) = \boldsymbol{G}(\boldsymbol{p}(\boldsymbol{\psi}), \boldsymbol{\psi}) \tag{5}$$

The sensitivity expression [25,24] is obtained by differentiating $\boldsymbol{F}$ with respect to each component of $\boldsymbol{\psi}$, ψ_i;

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}\psi_i} = \frac{\partial\boldsymbol{G}}{\partial\boldsymbol{p}} \cdot \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\psi_i} + \frac{\partial\boldsymbol{G}}{\partial\psi_i} \tag{6}$$

In Eq. (6) the terms $\partial\boldsymbol{G}/\partial\boldsymbol{p}$ and $\partial\boldsymbol{G}/\partial\psi_i$ are explicit, nevertheless the term $\mathrm{d}\boldsymbol{p}/\mathrm{d}\psi_i$ is implicitly given by Eq. (4). In the direct differentiation technique, this last term may then be calculated by differentiating equation (4), resulting in

$$\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{p}}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\psi_i} = \frac{\partial\boldsymbol{R}}{\partial\psi_i} \tag{7}$$

Eq. (8) is formally identical to Eq. (3), having the same tangent operator. Therefore, the evaluation of the sensitivity term requires only the formation of the pseudo-load vector for each design parameter ψ_i. This equation may be rewritten in the following form:

$$\boldsymbol{K}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\psi_i} = -\tilde{\boldsymbol{R}} \tag{8}$$

where $\boldsymbol{K}$ is the stiffness matrix and $-\tilde{\boldsymbol{R}}$ is the pseudo-load vector. In Section 5 it will be seen how these parameters are related to the primal problem and how they may be automatically constructed. Next, in Section 3 the particular features of the FEM formulation for crack propagation problems will be described.

3. Problem description

After a general reference to sensitivity analysis, the problem characteristics are particularised in this section. As referred, sensitivity analysis will be applied to crack propagation and therefore this section is mainly dedicated to the description of the associated kinematic quantities.

3.1. The XFEM

The XFEM is a highly flexible method which allows the description of discontinuities independently from the mesh. In the particular case of cracks, the standard approximation of the displacement field may be enriched as follows:

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{i=1}^{n} N_i\boldsymbol{u}_i + \sum_{j=1}^{n_{split}} N_j[H(\boldsymbol{x}) - H(\boldsymbol{x}_j)]\boldsymbol{a}_j + \sum_{k=1}^{n_{tip}} N_k R(\boldsymbol{x})[H(\boldsymbol{x}) - H(\boldsymbol{x}_k)]\boldsymbol{b}_k \tag{9}$$