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Sensitivity analysis of the variance contributions with respect to the distribution parameters by the kernel function

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ABSTRACT

Variance based sensitivity indices represent how the input uncertainty influences the output uncertainty. In order to identify how the distribution parameters of inputs influence the variance contributions, this work proposes the sensitivity of the variance contributions, which is defined by the partial derivative of the first-order variance contribution with respect to the distribution parameter. The proposed sensitivity can reflect how small variation of the distribution parameter influences the first-order variance contribution. By simplifying the partial derivative of the first-order variance contribution into the form of expectation via the kernel function, the proposed sensitivity can be seen as a by-product of the variance based sensitivity analysis without any additional output evaluations. For the classical quadratic responses, the proposed sensitivity can be derived analytically based on the integral form, while for the complex responses, the state dependent parameter (SDP) based method, which has been applied in the variance sensitivity analysis, can be employed to compute the proposed sensitivity. Several examples are used to demonstrate the correctness of the analytical solutions and the efficiency of the SDP based method.

1. Introduction

Development of probabilistic sensitivity analysis is frequently considered as an essential component of a probabilistic analysis [1]. Traditional sensitivity analysis (SA) can be classified into two groups: local SA and global SA [2]. Local SA usually investigates how small variation of the distribution parameter around the reference point changes the value of the output. One classical local SA is the derivative based SA which is defined as the derivative of response merits with respect to the distribution parameters of inputs. The main drawback of the derivative based SA is that it depends on the choice of the nominal point, while the main advantage of it is the low computational cost that is gained in turn [3].

Global SA studies how the uncertainty in the output of a computational model can be decomposed according to the input sources of uncertainty [4]. Contrary to the local SA, global SA explores the whole range of uncertainty of the model inputs by letting them vary simultaneously [5]. At present, a number of global sensitivity indices have been suggested, e.g. Helton and Saltelli proposed the non-parametric sensitivity indices (input–output correlation) [6,7], Sobol, Iman and Saltelli proposed the variance based sensitivity indices [7–9], Chun, Liu and Borgonovo proposed the moment independent sensitivity indices [10,11], Sobol and Kucherenko proposed the derivative based sensitivity indices and investigated their link with the variance based sensitivity indices [12–14]. In this work, we mainly investigate the variance based sensitivity indices which have been applied to design under uncertainty problems and are capable of identifying the contributions of any random variable. However, the variance based sensitivity indices are far more computationally demanding and various methods have

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been used to compute the variance based sensitivity indices, such as Sobol's method [15], the FAST method [16], and the meta-model based method [17]. Those methods all have their deficiencies, Sobol's method and the FAST method which are indeed computationally demanding for the engineering applications, and the meta-model based method cannot estimate the total variance based sensitivity indices if the interactions are higher than the second order in the ANOVA decomposition.

It is noticed that in the classical variance based SA, the influences of distribution parameters are not considered. If the variation of a distribution parameter can lead to the considerable changes to the variance contributions, the computational results of the variance based SA will be vulnerable and less reliable. Thus, analysts need to investigate the effects of the distribution parameters on the variance contributions in the variance based SA. In this work, we propose the sensitivity of distribution parameters on variance contributions which is defined by the partial derivative of the first-order variance contribution (FOVC) with respect to the distribution parameter. The proposed sensitivity can reflect how small variations of distribution parameters influence the FOVC.

In Ref. [1], Millwater has employed the kernel function which can be derived analytically for various distribution types to simplify the partial derivative based sensitivity of the first two moments and the failure probability, and then the partial derivative based sensitivity can be computed efficiently. Thus, in this work, we also employ the kernel function to simplify the proposed sensitivity first. Then for the simplified sensitivity, solutions for the classical quadratic polynomial without cross-terms response are derived analytically, while for the complex responses, the SDP based method, which is developed recently by Ratto [18] and Li [19], is employed to compute the proposed sensitivity.

The remainder of this work is organized as follows: Section 2 gives a brief review of the variance based SA. Section 3 first develops the sensitivity of distribution parameters on the FOVC, then the proposed sensitivity for the classical quadratic polynomial without cross-terms response is derived analytically. Some discussions for the effect of the kernel function and some computational considerations of the proposed sensitivity are given in Section 4. In Section 5, the SDP based method is employed to compute the proposed sensitivity. In Section 6, two numerical examples are first given to validate the correctness of the analytical solutions for the classical quadratic polynomial without cross-terms response and the efficiency of the SDP based method, then a simple cantilever with explicit response and a ten-bar structure with implicit response are employed to validate the reasonability of the proposed sensitivity. Finally, some conclusions are drawn in Section 7.

2. A brief review of the variance based sensitivity analysis

Consider a square integrable function $Y = g(\boldsymbol{X})$ defined in the hypercube H^n, where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ are n independent inputs. The method of variance based sensitivity indices developed by Sobol is based on ANOVA decomposition and there exists the following unique decomposition [8]:

$$g(\boldsymbol{X}) = g_0 + \sum_{i=1}^{n} g_i(X_i) + \sum_{1\le i<j\le n}^{n} g_{ij}(X_i, X_j) + \cdots + g_{1,2,\ldots,n}(X_1, X_2, \ldots, X_n) \tag{1}$$

where

$$\begin{aligned} g_0 &= E(Y) \\ g_i &= E(Y|X_i) - E(Y) \\ g_{ij} &= E(Y|X_i, X_j) - g_i - g_j - E(Y) \end{aligned} \tag{2}$$

$E(Y)$ is the expectation of the output, $E(Y|\cdot)$ is the conditional expectation of the output and the higher order items can be obtained analogously.

The variance of the output variable Y can thus be decomposed into:

$$V = \sum_{i=1}^{n} V_i + \sum_{1\le i<j\le n}^{n} V_{ij} + \cdots + V_{1,2,\ldots,n} \tag{3}$$

where V is the total variance, and

$$\begin{aligned} V_i &= V(Y_i) = V(E(Y|X_i)) \\ V_{ij} &= V(Y_{ij}) = V(E(Y|X_i, X_j)) - V(E(Y|X_i)) - V(E(Y|X_j)) \end{aligned} \tag{4}$$

are the first-order and second-order variance contributions of inputs, respectively.

In this approach, the FOVC of input X_i can be given in the form of integration:

$$V_i = V_{X_i}(E_{\boldsymbol{X}_{-i}}(Y|X_i)) = E\left((E(y|x_i) - E(y))^2\right) = \int_{R^1} (E(y|x_i) - E(y))^2 f_{X_i}(x_i)dx_i \tag{5}$$

where $E(y) = \int_{R^n} g(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}$ and $E(y|x_i) = \int_{R^{n-1}} g(\boldsymbol{x})f_{\boldsymbol{X}_{-i}}(\boldsymbol{x}_{-i})d\boldsymbol{x}_{-i}$ are the expectation and the conditional expectation of the output respectively. $f_{X_i}(x_i)$ is the marginal PDF of input X_i, $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint PDF of input vector $\boldsymbol{X}$ and $f_{\boldsymbol{X}_{-i}}(\boldsymbol{x}_{-i})$ is the joint PDF of all inputs except X_i. The FOVC can reflect the reduced portions of the output variance when the uncertainty of input X_i is eliminated. We look for the input that, once "discovered" with its true value and constrained at this value, would reduce the uncertainty of the output the most and, therefore, make the model inference more robust [20].