Nonlinear Analysis 103 (2014) 98–112

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Shape sensitivity analysis of the Hardy constant

Gerassimos Barbatis[a], Pier Domenico Lamberti[b,*]

[a] Department of Mathematics, University of Athens, 15784 Athens, Greece

[b] Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy

ARTICLE INFO

Article history:
Received 16 October 2013
Accepted 3 March 2014
Communicated by Enzo Mitidieri

MSC:
26D15
35P15
35P30

Keywords:
Hardy constant
Domain perturbation
Hadamard formula
Stability estimates

ABSTRACT

We consider the Hardy constant associated with a domain in the n-dimensional Euclidean space and we study its variation upon perturbation of the domain. We prove a Fréchet differentiability result and establish a Hadamard-type formula for the corresponding derivatives. We also prove a stability result for the minimizers of the Hardy quotient. Finally, we prove stability estimates in terms of the Lebesgue measure of the symmetric difference of domains.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $d_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$, $x \in \Omega$, and $p \in]1, \infty[$. If there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^p dx \geq c \int_\Omega \frac{|u|^p}{d_\Omega^p} dx, \quad \text{for all } u \in C_c^\infty(\Omega), \tag{1.1}$$

we then say that the L^p Hardy inequality holds in Ω. The best constant c for inequality (1.1) is called the L^p Hardy constant of Ω and we shall denote it by $H_p(\Omega)$. It is well-known that if Ω is regular enough then the L^p Hardy inequality is valid for all $p \in]1, \infty[$; moreover if Ω is convex, and more generally if it is weakly mean convex, i.e. if $\Delta d_\Omega \leq 0$ in the distributional sense in Ω, then $H_p(\Omega) = ((p-1)/p)^p$.

The study of inequality (1.1) has a long history which goes back to Hardy himself, see [1]. In the last twenty years there has been a growing interest in the study of Hardy inequalities, the existence and behavior of minimizers [2,3], improved inequalities [4,5], higher order analogues and other related problems.

The precise evaluation of $H_p(\Omega)$ for domains Ω that are not weakly mean convex is a difficult problem. There are only few examples of such domains for which $H_p(\Omega)$ is known and these are only for the case $p = 2$ and for very special domains Ω. Even the problem of estimating from below $H_p(\Omega)$ is difficult and most results again are for $p = 2$. One such result is the well known theorem by A. Ancona which states that $H_2(\Omega) \geq 1/16$ for all simply connected planar domains. We refer to [6,2,5,7–10] for more information on the Hardy constant.

* Corresponding author. Tel.: +39 0498271419; fax: +39 0498271499.
E-mail address: lamberti@math.unipd.it (P.D. Lamberti).

http://dx.doi.org/10.1016/j.na.2014.03.003
0362-546X/

In this paper we study the variation of $H_p(\Omega)$ upon variation of the domain Ω. This problem can be considered as a spectral perturbation problem. Indeed, if there exists a minimizer $u \in W_0^{1,p}(\Omega)$ for the Hardy quotient associated with (1.1) then u is a solution to the equation

$$-\Delta_p u = H_p(\Omega)\frac{|u|^{p-2}u}{d_\Omega^p} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. Domain perturbation problems have been extensively studied in the case of the Dirichlet Laplacian as well as for more general elliptic operators, such as operators satisfying other boundary conditions, higher order operators and operators with variable coefficients. We refer to the monographs [11,12] for an introduction to this topic. When studying such problems, there are broadly speaking two types of results: qualitative and quantitative. The former provide information such as continuity or analyticity, while the second involve stability properties, possibly together with related estimates. The relevant literature is vast, and we refer to [13–17] and references therein for more information; in particular, for the p-Laplacian we refer to [18–20].

In this paper we obtain both qualitative and quantitative results on the domain dependence of $H_p(\Omega)$. In Theorem 8, we assume that Ω is of class C^2 with $H_p(\Omega) < ((p-1)/p)^p$ and we establish the Fréchet differentiability of $H_p(\phi(\Omega))$ with respect to the C^2 diffeomorphism ϕ. In particular we provide a Hadamard-type formula for the Fréchet differential. For our proof we make an essential use of certain results of [2], where it was shown in particular that if $H_p(\Omega) < ((p-1)/p)^p$ then the Hardy quotient admits a positive minimizer u which behaves like d_Ω^α near $\partial\Omega$ for a suitable $\alpha > 0$. In fact, in Theorem 6 we also prove the stability of the minimizer u in $W_0^{1,p}(\Omega)$; this is of independent interest but is also used in the proof of Theorem 8.

We subsequently consider stability estimates for $H_p(\Omega)$. In Theorem 11 we prove under certain assumptions that the Hardy constant $H_p(\Omega)$ of a C^2 domain Ω is upper semicontinuous with respect to bi-Lipschitz transformations ϕ. In Theorem 12 we consider the stability of the Hardy constant when Ω is subject to a localized perturbation which transforms it to a domain $\tilde{\Omega}$. Assuming that both Ω and $\tilde{\Omega}$ are of class C^2 we obtain stability estimates for the L^p Hardy constant in terms of the Lebesgue measure of the symmetric difference $\Omega\triangle\tilde{\Omega}$. Estimates of this type have been recently obtained for eigenvalues of various classes of operators; we refer to [21,14,13] and references therein for more information.

We finally note that our results are new also for the linear case $p = 2$.

The paper is organized as follows. In Section 2 we introduce our notation and prove a general Lipschitz continuity result. Section 3 is devoted to the proof of differentiability results, the Hadamard formula and the stability of minimizers. In Section 4 we prove stability estimates in terms of the Lebesgue measure of the symmetric difference of the domains.

2. Preliminaries

Let Ω be a bounded domain (i.e. a bounded connected open set) in $\mathbb{R}^n$. Given $p \in]1, +\infty[$ we denote by $W_0^{1,p}(\Omega)$ the closure in the standard Sobolev space $W^{1,p}(\Omega)$ of the set of all smooth functions with compact support in Ω.

If $u \in W_0^{1,p}(\Omega),\ u \neq 0$, we then denote by $R_\Omega[u]$ the Rayleigh quotient

$$R_{\Omega,p}[u] = \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega \frac{|u|^p}{d_\Omega^p} dx},$$

and we set

$$H_p(\Omega) = \inf_{u \in W_0^{1,p}(\Omega), u\neq 0} R_{\Omega,p}[u]. \tag{2.1}$$

If $H_p(\Omega) > 0$ we then say that the L^p Hardy inequality is valid on Ω.

It is well known that if Ω has a Lipschitz continuous boundary then $0 < H_p(\Omega) \leq ((p-1)/p)^p$ and it has been proved in [2,3] that if Ω is of class C^2 then there exists a minimizer u in (2.1) if and only if $H_p(\Omega) < ((p-1)/p)^p$; moreover, such minimizer is unique up to a multiplicative constant, can be chosen to be positive and there exists $c > 0$ such that

$$c^{-1} d_\Omega(x)^\alpha \leq u(x) \leq c d_\Omega(x)^\alpha, \quad x \in \Omega, \tag{2.2}$$

where $\alpha > (p-1)/p$ is the largest solution to the equation

$$(p-1)\alpha^{p-1}(1-\alpha) = H_p(\Omega). \tag{2.3}$$

Given a Lipschitz map $\phi : \Omega \to \phi(\Omega)$ we define $\operatorname{Lip}(\phi) = \|\nabla\phi\|_{L^\infty(\Omega)}$. For $L > 0$ we define the uniform class of bi-Lipschitz maps

$$\mathrm{bLip}_L(\Omega) = \{\phi : \Omega \to \phi(\Omega) : \phi,\ \phi^{(-1)} \text{ are Lipschitz continuous and } \operatorname{Lip}(\phi),\ \operatorname{Lip}(\phi^{(-1)}) \leq L\}.$$

In the sequel we shall often use the fact that $H_p(\phi(\Omega))$ depends continuously on ϕ. In fact, we can prove the following Lipschitz continuity result.

Note that in the proof of the following proposition as well as in the proofs of other statements in the sequel, by c, c_1 etc. we shall denote constants, the value of which may change from line to line.