Finance Research Letters 8 (2011) 52–58

Contents lists available at ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

A note on reward-risk portfolio selection and two-fund separation

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A R T I C L E I N F O

Article history:
Received 3 November 2010
Accepted 22 November 2010
Available online 16 December 2010

JEL classification:
G11
D81
D14
C61

Keywords:
Two-fund separation
Reward-risk preferences

A B S T R A C T

This paper presents a general reward-risk portfolio selection model and derives sufficient conditions for two-fund separation. In particular we show that many reward-risk models presented in the literature satisfy these conditions.

1. Introduction

The modern portfolio theory of Markowitz (1952) is a rich source of intuition and also the basis for many practical decisions. Mean-variance agents may differ with respect to the degree they are willing to trade off mean against variance but all will choose from the set of efficient portfolios, those which maximize mean given a constraint on variance. Moreover, under certain conditions, the mean-variance model of portfolio selection leads to two-fund separation (Tobin, 1958), i.e., all agents hold a combination of the *same* portfolio of risky assets combined with the risk-free asset. Two-fund separation greatly simplifies the advice one should give to a heterogenous set of agents since the proportion of risky assets in the optimal portfolio is independent from agent's risk aversion. Moreover, it implies a simple asset pricing structure in which a single risk factor explains the rewards agents get in equilibrium.

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1544-6123/$ - see front matter
doi:10.1016/j.frl.2010.11.003

We derive sufficient conditions for two-fund separation in a general reward-risk model, where agents' preferences are assumed to be increasing functions of a reward measure and decreasing functions of a risk measure. We show that two-fund separation holds if reward and risk measures can be transformed by means of strictly increasing functions into positive homogeneous, translation invariant or translation equivariant functionals. In this case, the efficient frontier is a straight line in the transformed reward-risk diagram.

Several reward and risk measures introduced in the literature satisfy the conditions for two-fund separation. Mean and variance, semi-variance (Markowitz, 1959), lower partial moments (Bawa and Lindenberg, 1977; Fishburn, 1977; Harlow and Rao, 1989), the Gini measure (Yitzhaki, 1982), general deviation measures (Rockafellar et al., 2006), etc. Since many of these measures are defined based only on few principles of rationality, it follows that two-fund separation is a common property to many rational mean-risk models, including even those which are consistent with second order stochastic dominance (De Giorgi, 2005). This result is surprising because strong conditions on agents' utility functions are needed in order to obtain two-fund separation within expected utility theory (Cass and Stiglitz, 1970).

In Section 2 we introduce the general reward-risk model and derive our main result. Examples are discussed in Section 3. All proofs are given in Appendix A.

2. General reward-risk model and two-fund separation

We consider a two-period economy. The set S of states of the world in the second period is endowed with a sigma-algebra Σ. An element $s \in S$ is an individual state of the world, while $A \in \Sigma$ is an event. A random variable on (S, Σ) is a real-valued function $X : S \to \mathbb{R}$ such that $X^{-1}(J) \in \Sigma$ for all intervals $J \subset \mathbb{R}$. The space of random variables on (S, Σ) is denoted by $\mathscr{L}_0(S, \Sigma)$. We do not assume existence of a probability measure on (S, Σ), since our conditions for two-fund separation do not require any assumption on the *distribution* of random variables.

There are $K + 1$ assets with random payoffs $A_k \in \mathscr{L}_0(S, \Sigma)$ and prices q_k, $k = 0, \ldots, K$. Asset 0 is the risk-free asset with $A_0 = 1$, $q_0 > 0$, and gross return $R_0 = 1/q_0$. We assume that assets can be traded without restrictions: the marketed subspace is denoted by $\mathscr{X} = \left\{ \sum_{k=0}^{K} \theta_k A_k | (\theta_0, \ldots, \theta_K)' \in \mathbb{R}^{K+1} \right\}$. An element $X \in \mathscr{X}$ is called a portfolio. For $X = \sum_{k=0}^{K} \theta_k A_k \in \mathscr{X}$ we denote by $q(X) = \sum_{k=0}^{K} \theta_k q_k$ the price of portfolio X.

We introduce two real-valued functions $\mu : \mathscr{X} \to \mathbb{R}$ and $\rho : \mathscr{X} \to \mathbb{R}$. We call μ a *reward* measure and ρ a *risk* measure. It is plausible to assume that $\mu(X) \geqslant \mu(Y)$ and $\rho(X) \leqslant \rho(Y)$ when $X(s) \geqslant Y(s)$ for all $s \in S$. We do not make these monotonicity assumptions here since they are not necessary for our main result and they would exclude from our framework the mean-variance model (variance violates monotonicity). The basic properties of reward and risk measures we consider are the following.

Definition 1. Let $\zeta : \mathscr{X} \to \mathbb{R}$ be a real-valued function on $\mathscr{X}$ (ζ is a reward or a risk measure). We say that ζ is:

(i) (*positive*) homogeneous of degree γ *if*

$$\zeta(\kappa X) = |\kappa|^{\gamma} \zeta(X),$$

for all $\kappa \in \mathbb{R}$ ($\kappa \geqslant 0$) and $X \in \mathscr{X}$,

(ii) translation invariant *if*

$$\zeta(X + a) = \zeta(X)$$

for all $a \in \mathbb{R}$ and $X \in \mathscr{X}$,

(iii) translation equivariant *if*

$$\zeta(X + a) = \zeta(X) + a$$

for all $a \in \mathbb{R}$ and $X \in \mathscr{X}$, or

$$\zeta(X + a) = \zeta(X) - a$$

for all $a \in \mathbb{R}$ and $X \in \mathscr{G}$. In the first case, we say that ζ is *positive* translation equivariant, while in the second case it is *negative* translation equivariant.