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A numerical method for the expected penalty–reward function in a Markov-modulated jump–diffusion process

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ABSTRACT

A generalization of the Cramér–Lundberg risk model perturbed by a diffusion is proposed. Aggregate claims of an insurer follow a compound Poisson process and premiums are collected at a constant rate with additional random fluctuation. The insurer is allowed to invest the surplus into a risky asset with volatility dependent on the level of the investment, which permits the incorporation of rational investment strategies as proposed by Berk and Green (2004). The return on investment is modulated by a Markov process which generalizes previously studied settings for the evolution of the interest rate in time. The Gerber–Shiu expected penalty–reward function is studied in this context, including ruin probabilities (a first-passage problem) as a special case. The second order integro-differential system of equations that characterizes the function of interest is obtained. As a closed-form solution does not exist, a numerical procedure based on the Chebyshev polynomial approximation through a collocation method is proposed. Finally, some examples illustrating the procedure are presented.

1. Introduction

The risk process presented by Gerber (1970) extends the classical model of risk theory introducing a Brownian diffusion. The total claims follow a compound Poisson process $\{X_t, t \geq 0\}$ with Lévy measure $\lambda f(x)\mathrm{d}x$, λ being the intensity of arrivals and f the density of jumps. The collection of premiums is driven by a Wiener process W_t^c independent of X_t with drift c and volatility σ, thus the perturbed risk process with initial surplus u is given by

$$\mathrm{d}R_t = c\mathrm{d}t + \sigma \mathrm{d}W_t^c - \mathrm{d}X_t, \qquad R_0 = u. \tag{1}$$

This process has been considered by Dufresne and Gerber (1991) where a defective renewal equation was derived for the probability of ruin $\psi(u) = \Pr(\tau < \infty)$ where $\tau = \inf\{t \geq 0 : R_t < 0\}$. A review of the research on this type of process can be found in Asmussen and Albrecher (2010), Chapter 11. Generalizations of the model are treated in Li and Garrido (2005), Sarkar and Sen (2005), and Morales (2007), whereas Ren (2005) gives explicit formulae to calculate the ruin probability and related quantities for phase-type distributed claims.

Let us now allow the insurer to invest the reserves U_t into an asset with time-dependent Markov-modulated return rate (drift) Δ_t and volatility $\kappa(U_t)$, that possibly depends on the amount invested U_t, driven by a Wiener process W_t^I independent of the risk process R_t

$$\mathrm{d}U_t = (\Delta_t \mathrm{d}t + \kappa(U_t)\mathrm{d}W_t^I)U_t + dR_t, \qquad U_0 = R_0 = u. \tag{2}$$

The drift parameter Δ_t is governed by a finite state homogeneous Markov process with state space $\{\delta_1, \ldots, \delta_n\}$, intensity matrix $Q = (q_{ij})_{n\times n}$ and initial state δ_i. For example, Δ_t can be used to model the risk free rate announced by a central bank that evolves according to the Markov process by, for instance, 25 basis point jumps. The state space would be in this case e.g.,

1.00%, 1.25%, 1.50%, 1.75%, 2.00%, ..., 9.00%.

This environment offers considerable versatility in capturing the evolution of interest rates since any diffusion model to forecast the yield curve can be approximated arbitrarily well by continuous time Markov chains; see Kushner and Dupuis (1992). Variation of the volatility according to the size of the funds invested is justified, for example, by Berk and Green (2004) as an implication of their study of the performance of mutual funds and resulting rational

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capital flows. A particular shape of κ suggested in the cited paper, $\kappa(u) = \frac{\sigma_r}{\sqrt{u}}$, yields a surplus process in the form of an affine diffusion that was studied by Avram and Usabel (2008) in this context. Many practical ideas support a fund-dependent volatility, for instance the possibility to obtain more efficient portfolios, due to transaction costs, when more money is available. Model (2) is a generalization of the process considered most frequently in the literature where the return rate and the volatility are constant in time, $\Delta_t = \delta$, $\kappa(\cdot) = \sigma_r$, like in Paulsen (1993), Paulsen and Gjessing (1997), Wang (2001), Ma and Sun (2003), Gaier and Grandits (2004), Grandits (2005), Cai and Yang (2005) and Wang and Wu (2008).

The stochastic differential equation (2) can be arranged into

$$\mathrm{d}U_t = (c + \Delta_t U_t)\mathrm{d}t + \sqrt{\sigma^2 + \kappa^2(U_t)U_t^2}\,\mathrm{d}W_t - \mathrm{d}X_t \tag{3}$$

with initial condition $(U_0, \Delta_0) = (u, \delta_i)$. The expected penalty–reward function (see Gerber and Landry (1998)) is introduced

$$\phi_t^i(u) = \mathrm{E}[\pi(U_\tau)\mathbb{I}(\tau \le t) + P(U_t)\mathbb{I}(\tau > t) \mid U_0 = u, \Delta_0 = \delta_i] \tag{4}$$

where $\tau = \inf\{s \ge 0 : U_s < 0\}$. If ruin occurs before the time horizon t, the penalty $\pi(U_\tau)$ applies to the overshoot U_τ at the ruin. Otherwise, the reward function $P(U_t)$ applies to the reserves at time t. The concept of the expected penalty–reward function presented in Gerber and Shiu (1997) and Gerber and Shiu (1998) is a quite general framework comprising several quantities of interest as a special case, such as the time to ruin, the amount at and immediately prior to ruin or survival probabilities.

For further analysis the smoothed version of the function $\phi_t^i(u)$ will be considered, namely its Laplace–Carson transform in time defined as

$$\Upsilon_\alpha^i(u) = \int_0^\infty \alpha \mathrm{e}^{-\alpha t}\phi_t^i(u)\mathrm{d}t.$$

Further, letting H_α be an exponentially distributed random variable with parameter α, the former expression may be viewed as a penalty–reward function with an exponentially killed time horizon (see expression (6) in Avram and Usabel (2008))

$$\begin{aligned}\Upsilon_\alpha^i(u) &= \int_0^\infty \alpha \mathrm{e}^{-\alpha t}\phi_t^i(u)\mathrm{d}t = \mathrm{E}(\phi_{H_\alpha}^i(u)) \\ &= \mathrm{E}(\pi(U_\tau)\mathbb{I}(\tau \le H_\alpha) \\ &\quad + P(U_{H_\alpha})\mathbb{I}(\tau > H_\alpha) \mid U_0 = u, \Delta_0 = \delta_i)\end{aligned} \tag{5}$$

where the last equality comes from substituting the definition of $\phi_t^i(u)$, in (4).

The function $\Upsilon_\alpha^i(u)$ is analytically more tractable than the original function while, at the same time, retains a probabilistic interpretation as a penalty–reward function considering an exponential random time horizon H_α.

The results in this paper are organized as follows: in Section 2 an integro-differential system that characterizes the function of interest $\Upsilon_\alpha^i(u)$ is derived and the existence of the solution discussed. In Section 3 a numerical method to approximate the solution of the system via Chebyshev polynomials is considered and Section 4 offers some numerical illustrations.

2. Integro-differential system

This section presents further treatment of the transformed expected penalty–reward function defined by (5). The function $\Upsilon_\alpha^i(u)$ is dependent on the initial reserves $U_0 = u$ and the starting return rate $\Delta_0 = \delta_i$. Since the process driving the return rate Δ_t has a finite state space, the number of initial conditions is also finite. Therefore, one can consider the set of functions $\Upsilon_\alpha(u) = (\Upsilon_\alpha^1(u), \Upsilon_\alpha^2(u), \ldots, \Upsilon_\alpha^n(u))$, each corresponding to different starting return rate from the state space $\{\delta_1, \ldots, \delta_n\}$. Below, a Volterra integro-differential system of equations for the functions $\Upsilon_\alpha^1(u), \Upsilon_\alpha^2(u), \ldots, \Upsilon_\alpha^n(u)$ is derived and, applying the result of Le and Pascali (2009), sufficient conditions for the existence of the solution are established.

Theorem 2.1. *For all $\alpha \ge 0$, functions $\Upsilon_\alpha^i : [0, \infty) \to \mathbb{R}$ defined in (5) satisfy the following system of integro-differential equations*
For $i = 1, \ldots, n$

$$\begin{aligned}&\frac{1}{2}(\sigma^2 + u^2\kappa^2(u))\frac{\mathrm{d}^2}{\mathrm{d}u^2}\Upsilon_\alpha^i(u) + (c + \delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\Upsilon_\alpha^i(u) \\ &\quad + \sum_{j=1}^n q_{ij}\Upsilon_\alpha^j(u) - (\alpha + \lambda)\Upsilon_\alpha^i(u) \\ &\quad + \lambda \int_0^u \Upsilon_\alpha^i(u - x) f(x)\,\mathrm{d}x \\ &\quad + \alpha P(u) + \lambda \int_u^\infty \pi(u - x) f(x)\,\mathrm{d}x = 0.\end{aligned} \tag{6}$$

Given that $\lim_{u\to\infty} P(u)$ exists, $\sigma > 0$ and assuming positive security loading for the reserve process (2), the boundary conditions of the system are

$$\Upsilon_\alpha^i(0) = \pi(0-) \tag{7}$$

$$\lim_{u\to\infty} \Upsilon_\alpha^i(u) = \lim_{u\to\infty} P(u) \equiv P(\infty).$$

Moreover, if $f \in C^2[0, \infty)$, $P(u)$ and $\kappa(u)$ are continuous for $u \ge 0$ and $\pi(u)$ integrable, then the system of Eq. (6) has a solution $\Upsilon_\alpha^i \in C^2[0, \infty)$, $i = 1, \ldots, n$.

Proof. First, a straightforward application of Ito's lemma yields the infinitesimal generator of the process U_t, which applied to the functions $\phi_t^i(u)$, $i = 1, \ldots, n$ defined by (4), yields

$$\begin{aligned}\mathcal{A}\phi_t^i(u) &= \frac{1}{2}(\sigma^2 + u^2\kappa^2(u))\frac{\mathrm{d}^2}{\mathrm{d}u^2}\phi_t^i(u) \\ &\quad + (c + \delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\phi_t^i(u) + \sum_{j=1}^n q_{ij}\phi_t^j(u) \\ &\quad + \lambda \int_0^\infty (\phi_t^i(u - x) - \phi_t^i(u)) f(x)\,\mathrm{d}x.\end{aligned}$$

Functions $\phi_t^i(u)$ satisfy the Fokker–Planck equation (see e.g. Risken (1996))

$$\mathcal{A}\phi_t^i(u) - \frac{\partial \phi_t^i(u)}{\partial t} = 0 \tag{8}$$

with boundary conditions

$$\phi_0^i(u) = P(u) \qquad u > 0 \tag{9a}$$

$$\phi_t^i(u) = \pi(u) \qquad u < 0 \text{ and } t \ge 0 \tag{9b}$$

for each $i = 1, 2, \ldots, n$. Using (9b) the following holds

$$\int_0^\infty \phi_t^i(u - x) f(x)\,\mathrm{d}x = \int_0^u \phi_t^i(u - x) f(x)\,\mathrm{d}x + \int_u^\infty \pi(u - x) f(x)\,\mathrm{d}x. \tag{10}$$

Substituting the infinitesimal generator and (10) into the Fokker–Planck equation yields

$$\frac{1}{2}(\sigma^2 + u^2\kappa^2(u))\frac{\mathrm{d}^2}{\mathrm{d}u^2}\phi_t^i(u) + (c + \delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\phi_t^i(u)$$