General quadratic performance analysis and synthesis of differential algebraic equation (DAE) systems

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Abstract

In this paper, control of linear differential-algebraic-equation systems, subject to general quadratic constraints, is considered. This setup, especially, includes the H_∞ control problem and the design for strict passivity. Based on linear matrix inequality (LMI) analysis conditions, LMI synthesis conditions for the existence of linear output feedback controllers are derived by means of a linearizing change of variables. This approach is constructive: a procedure for the determination of controller parameterizations is given on the basis of the solution of the LMI synthesis conditions. A discussion of the possible applications of the presented results concludes the paper. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Differential-algebraic equation (DAE) systems (sometimes also referred to as singular, semistate or descriptor systems) describe a broad class of systems which are not only of theoretical interest but also have great practical significance. Models of chemical processes for example typically consist of differential equations describing the dynamic balances of mass and energy while additional algebraic equations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, etc. [7]. In mechanical engineering DAE system descriptions result from holonomic and non-holonomic constraints [16]. Also in electronics and even in economics DAE descriptions are encountered [9].

DAE systems are able to describe system behaviors that cannot be captured by "non-DAE" systems (i.e. systems governed only by differential equations) [1]. Therefore, index reduction techniques (i.e. reduction of a DAE system to an ODE system) necessarily are connected to a loss of information for high index systems. Due to this fact much work has been focused on analysis and design techniques for linear DAE systems in recent years (see [8] for an overview). Even for index one DAE systems (i.e. DAE systems, which are equivalent to

an ODE system) it is sometimes tedious or numerically not reliable to use the inversion of the algebraic equations in order to incorporate ODE based controller computation methods. This is especially the case for problem descriptions in chemical process control, where it is not uncommon to encounter much more algebraic equations than differential equations (e.g. in distillation control). Also for ODE process models an ODE based controller computation may not be the natural method of choice: the actual control problem, typically, is given by the process model *plus* some weighting systems or filters *plus* the algebraic couplings between these systems, i.e. as a DAE system.

Quite recently LMI based analysis and synthesis methods have been introduced to DAE control problems, (but so far restricted to H_2- and H_∞-problems) [10,19,13]. In this paper we consider the LMI approach to the generalized quadratic performance (GQP) control problem. In [15] this problem is solved for non-DAE systems. The idea is to control a generalized linear plant such that the closed loop transfer function $G_{cl} : \mathbf{w} \in H_2 \mapsto \mathbf{z} \in H_2$ is internally stable and such that the general quadratic constraint

$$\int_0^T \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{w}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} U_P & W_P \\ W_P^{\mathrm{T}} & V_P \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{w}(t) \end{bmatrix} \mathrm{d}t \ll 0, \qquad (1)$$

is imposed on the external input and output functions $\mathbf{w}(\cdot)$ and $\mathbf{z}(\cdot)$ respectively. Here the notation "$\ll 0$"

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means: for a given quadratic scalar function $Q(w,z)$, $Q(w, z) \ll 0$ is defined as $\exists \epsilon > 0 : Q(w, z) \leq -\epsilon w^T w$ for all w. Analogously $\int_0^T Q(w(t)), z(t)) dt \ll 0$ means that $\int_0^T Q(w(t)), z(t)) dt \leq -\epsilon \int_0^T w(t)w(t)) dt$ holds for all $w(\cdot) \in L_2$ and some fixed $\epsilon > 0$.

The rather general GQP problem contains some important control problems as a special case if the objective parameters $U_P \geq 0$, $V_P = V_P^T$, and W_P are chosen accordingly. For example

- the H_∞ constraint $||G_{cl}||_\infty < \gamma$, if U_P, V_P, and W_P are specified as $U_P = \frac{1}{\gamma} I$, $V_P = -\gamma I$, $W_P = 0$;
- the strict passivity constraint $G_{cl}(j\omega) + G_{cl}(j\omega)^* > 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$, when U_P, V_P, W_P are chosen as $U_P = 0$, $V_P = 0$, $W_P = -I$;
- sector constraints of the form

$$\int_0^T (\mathbf{z}(t) - \alpha \mathbf{w}(t))^T (\mathbf{z}(t) - \beta \mathbf{w}(t)) dt \ll 0 \tag{2}$$

for $U_P = I$, $V_P = -\alpha\beta I$, $W_P = -\frac{1}{2}(\alpha + \beta)I$.

The key to the synthesis problem is a modified version of the linearizing change of variables approach used in the corresponding case for ODE systems [15]. In contrast to a previous paper [11] we do not assume the DAE description to be in semi-explicit form. Especially, it is possible to include the standard ODE result without any additional fall differentiation. The paper is structured as follows: in the next section the necessary background on linear DAE systems is given and we discuss the generalization of the "internal stability" concept to DAE systems. Subsequently a LMI analysis result for general quadratic performance is given. By means of this result it is possible, for a given controller, to efficiently decide, whether or not a closed loop system in DAE form meets the performance requirements. Also the structure of the LMI solution connected to this problem will be examined. Based on these results a direct treatment of the synthesis problem in the next section is possible: with the controller being unknown, the analysis result formally becomes a nonlinear matrix inequality. However, the presented linearizing change of variables approach reveals, that the problem can be reduced to a strict LMI problem. We consider the computational implications of the derived synthesis result and finally discuss the possible range of applications.

2. Background on linear DAE descriptions

We consider input–output descriptions in DAE form

$$E\dot{\xi}(t) = A\xi(t) + Bw(t), \quad t \geq 0, \xi(0^-) = \Xi_0^-$$

$$z(t) = C\Xi(t) + Dw(t). \tag{3}$$

Here $\Xi(t) \in \mathbb{R}^{n_\xi}$ denote the generalized state variables, $w(t) \in \mathbb{R}^{n_w}$ the external input variables, and $z(t) \in \mathbb{R}^{n_z}$ the external output variables. E, A, B, C, D are constant system matrices of appropriate dimensions with E being a possibly singular $n_\xi \times n_\xi$ matrix with $n_\xi \geq \text{rank}(E) =: r$. In contrast to standard linear systems with $E = I$ (3) with $\text{rank}(E) < n_\xi$ may have no solution, one solution, or even multiple solutions. In general the solutions exhibit impulsive behavior (i.e. are *generalized* solutions [2]) even if the inputs $w(\cdot)$, $u(\cdot)$ are continuous [1]. A necessary and sufficient condition for the existence and uniqueness of a solution is, that the pencil $sE - A$ is *regular*, i.e. $\det(sE - A) \not\equiv 0$ [1]. Regular DAE systems are termed *stable* if $\{s | s \in \mathbb{C}, \det(sE - A) = 0\} \subset \mathbb{C}^-$ [1]. If the pencil $sE-A$ is *singular*, i.e. $\det(sE-A) \not\equiv 0$ it can be shown [6] that the unforced ($w(\cdot) = \mathbf{0}, u(\cdot) = \mathbf{0}$) DAE system admits non-trivial solutions to the homogeneous initial value problem. Therefore the following substitute of the term "internal stability" seems natural [10].

Definition 2.1. A DAE system is said to be admissible if it is regular, stable, and has no impulsive solutions.[1]

3. General quadratic performance: analysis LMIs

An LMI based sufficient condition for admissible DAE systems (E, A, B, C, D) [abbreviation of (3)] which admit a general quadratic performance bound is given in the following proposition:

Proposition 3.1. *The DAE system (E, A, B, C, D) is admissible and has general quadratic performance if there exists a matrix X such that the linear matrix inequalities*

$$E^T X = X^T E \geq 0 \tag{4}$$

$$\begin{bmatrix} A^T X + X^T A & X^T B \\ B^T X & 0 \end{bmatrix} +$$
$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} U_P & W_P \\ W_P^T & V_P \end{bmatrix} \tag{5}$$
$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0$$

hold true.

Proof. Assume (4) and (5) hold true for some matrix X. Define $\mathcal{V}(\Xi) := \Xi^T E^T X \Xi$. Differentiation along trajectories of (3) renders

$$\frac{d}{dt} v(\Xi(t)) = \dot{\Xi}^T(t)E^T X\Xi(t) + \xi^T(t)X^T E\dot{\Xi}(t)$$

$$= \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} A^T X + X^T A & X^T B \\ B^T X & 0 \end{bmatrix} \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix}$$

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[1] In view of the Weierstrass canonical form [6] of a DAE system we frequently use the term "index one system" in stead of "regular system without impulsive solution".