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A convex approach to robust $\mathcal{H}_2$ performance analysis☆

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Abstract

In this paper, we consider the problem of assessing worst-case $\mathcal{H}_2$ performance for MIMO systems and we give an LMI based sufficient condition for robust performance under LTI (not necessarily causal) model uncertainty, having the same complexity as $\mathcal{H}_\infty$ conditions for the same problem. In addition, we show that this condition is indeed necessary and sufficient for MISO and SIMO systems under a class of LTI uncertainty, and for MIMO plants under (arbitrarily slow) LTV uncertainty. © 2002 Published by Elsevier Science Ltd.

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1. Introduction

$\mathcal{H}_2$ control theory is appealing since there is a well established connection between the performance index being optimized and performance requirements encountered in practical situations. Moreover, the resulting controllers are easily found by solving two Riccati equations, and in the state-feedback case exhibit good robustness properties (Anderson & Moore, 1990). However, as the classical paper (Doyle, 1978) established, these margins vanish in the output feedback case.

Following this paper, several attempts were made to incorporate robustness into the $\mathcal{H}_2$ framework (Stein & Athans, 1987; Zhang & Freudenberg, 1990). More recently these efforts led to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem (Bernstein & Haddad, 1989; Zhou, Glover, Bodenheimer, & Doyle, 1994; Kaminer, Khargonekar, & Rotea, 1993; Sznaier, 1994; Scherer, 1995; Chen & Wen, 1995), where the resulting controller guarantees optimal performance for the *nominal* plant and stability against LTI dynamic uncertainty. While these results represent significant progress towards obtaining robust $\mathcal{H}_2$ controllers, they suffer from the fact that only nominal performance is guaranteed. Moreover, the resulting controllers have potentially high order.

Robust $\mathcal{H}_2$ performance under non-causal, non-linear time varying perturbations was analyzed in (Stoorvogel, 1993). More recently, both state-space (Feron, 1997) and frequency domain (Paganini, 1995a, b) convex upper bounds on the worst case $\mathcal{H}_2$ norm have been proposed. The state-space based bound, obtained using dynamic stability multipliers), is appealing since it takes into account, to some extent, causality. However, in order to obtain tractable problems, these multipliers must be restricted to the span of some basis, selected a-priori. Moreover, the complexity of this basis is limited by the fact that the computational complexity of the resulting LMI problem grows roughly as the 10th power of the state dimension (Paganini & Feron, 1999). On the other hand, while the frequency-domain based methods (Paganini, 1995a, b) cannot impose causality, they lead to simple LMI based conditions. Unfortunately, as shown by Sznaier and Tierno (2000) both the time and frequency-domain based bounds can be conservative by a factor of $\sqrt{m}$, where m denotes the dimensions of the exogenous input, even for very simple plants.

In this paper, we consider the problem of assessing worst case $\mathcal{H}_2$ performance under both LTI and LTV uncertainty. The main result of the paper provides sufficient conditions for robust $\mathcal{H}_2$ performance in the presence of LTI uncertainty. Further, these conditions are necessary and sufficient

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for a class of SIMO and MISO systems or for MIMO systems subject to (arbitrarily slowly) time varying perturbations.

For simplicity, in the sequel all the derivations are carried out in the discrete-time case. However, the formulae apply as well to continuous-time systems with minimal modifications.

2. Preliminaries

2.1. Notation and definitions

By $\mathscr{H}_2$ we denote the space of complex valued matrix functions $G(\lambda)$ with analytic continuation in $|\lambda| < 1$ and square integrable on the unit disk, equipped with the usual $\mathscr{H}_2$ norm

$$\|G\|_2^2 \doteq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Trace}[G(e^{j\omega})G(e^{j\omega})^*]\,d\omega.$$

Given two matrices M and Δ of compatible dimensions we denote by $M \star \Delta$ the upper LFT $\mathscr{F}_u(M,\Delta)$, i.e.

$$M \star \Delta = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}.$$

Let $\mathscr{L}(\ell^2)$ denote the set of linear bounded operators in ℓ^2. In the sequel we will consider the following set of structured bounded operators in $\mathscr{L}(\ell^2)$:

$$\mathscr{B}\mathbf{\Delta} = \{\Delta \in \mathscr{L}(\ell^2)\colon \Delta = \mathrm{diag}[\delta_1 I_{r_1},\ldots,\delta_S I_{r_S}, \Delta_{S+1},\ldots,\Delta_{S+F}],\ \|\Delta\|_{\ell^2\to\ell^2} \leqslant 1\}.$$

The subsets of $\mathscr{B}\mathbf{\Delta}$ formed by linear time invariant, causal linear time invariant, linear time varying and (arbitrarily) slowly linear time varying operators will be denoted by $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$, $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}_{\mathrm{causal}}$, $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}$ and $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$,[1] respectively. For ease of notation we also introduce a set of *constant* complex matrices having a structure similar to that of the operators in $\mathscr{B}\mathbf{\Delta}$:

$$\mathscr{B}\mathbf{\Delta}_m = \{\Delta \in C^{n\times n}\colon \Delta = \mathrm{diag}[\delta_1 I_{r_1},\ldots,\delta_S I_{r_S}, \Delta_{S+1},\ldots,\Delta_{S+F}],\ \bar{\sigma}(\Delta) \leqslant 1\},$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value. Finally, we will also make use of the following set of scaling matrices which commute with the elements in $\mathscr{B}\mathbf{\Delta}$:

$$\mathbf{X} = \{X\colon X = \mathrm{diag}[X_1,\ldots,X_S, x_{S+1}I_{m_1},\ldots,x_{S+F}I_{m_F}],\ X = X^*\}.$$

2.2. The $\mathscr{H}_2$ norm for LTV systems

In this paper, we are interested in analyzing the worst case $\mathscr{H}_2$ norm of the interconnection shown in Fig. 1,

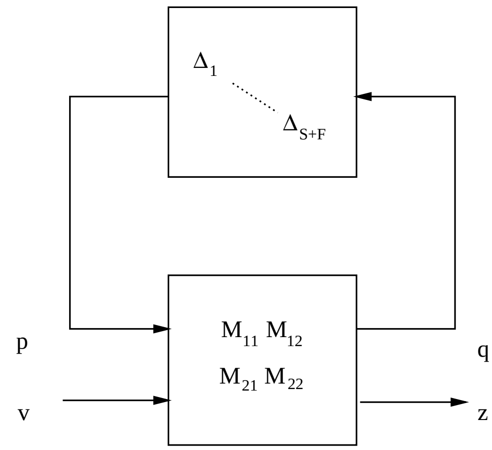

Fig. 1. Setup for robust $\mathscr{H}_2$ analysis.

where the nominal plant M is finite-dimensional, linear time invariant (FDLTI) and where the structure uncertainty $\mathbf{\Delta} = \mathrm{diag}\{\Delta_1,\Delta_2,\ldots,\Delta_{S+F}\}$ belongs to one of the classes discussed above. While this problem is well defined if $\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}_{\mathrm{causal}}$, extending it to the other cases requires an appropriate definition of the $\mathscr{H}_2$ norm for LTV systems.

Several such definitions have been proposed (see for instance Paganini & Feron, 1999). These definitions can essentially be divided into the following three groups: (i) energy of the impulse response, added (or averaged) over the input direction, (ii) a stochastic interpretation based on the covariance of the output due to Gaussian white noise, and (iii) a deterministic approach based on considering the $\mathscr{H}_2$ norm as an induced norm from a subset of ℓ^2 to ℓ^2, considering for instance either the subset of ℓ^2 formed by signals with unity spectral density or by signal "white up to a small quantity η" (Paganini, 1995a). It is well known that these definitions coincide for LTI systems, but do not do so in the LTV case. In pursuing the extension of (ii) to the LTV case, care must be exercised since the output to stationary noise may no longer be stationary. This leads to two different interpretations based on whether the average or worst case output variance are considered (Paganini & Feron, 1999). Both approaches (i) and (iii) extend naturally from the LTI to the (N)LTV case. Note in passing that for the case of LTI systems (i) and (iii) actually coincide, since the impulse is the worst case signal among both sets. The motivation for using the "energy of the impulse response" definition is less clear in the case of (N)LTV systems, since here this may no longer be the case. On the other hand, as pointed out by Paganini (1995a), using an induced norm approach allows for bringing to bear to the problem powerful methods originally developed in the context of $\mathscr{H}_\infty$ control. Moreover, as we show in the sequel, surprisingly, approaches (i) and (iii) also coincide in the LTV case, i.e. the worst-case signal over the set of signals in the unit spectral density ball can always be taken to be an impulse.

[1] In rigor $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ is a class containing all the LTV operators with variation slower than a given $v > 0$, i.e. $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}_v \doteq \{\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}\colon \|\lambda\Delta - \Delta\lambda\| \leqslant v\}$, where λ denotes the unit delay operator. In the sequel, for notational simplicity and with a slight abuse of notation we will drop the subscript v.