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A decision-theoretic foundation for reward-to-risk performance measures

Frank Schuhmacher [a,*], Martin Eling [b]

[a] University of Leipzig, Grimmaische Straße 12, 04109 Leipzig, Germany
[b] University of St. Gallen, Kirchlistrasse 2, 9010 St. Gallen, Switzerland

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ABSTRACT

In this paper we prove that partial-moments-based performance measures (e.g., Omega, Kappa, upside-potential ratio, Sortino–Satchell ratio, Farinelli–Tibiletti ratio), value-at-risk-based performance measures (e.g., VaR ratio, CVaR ratio, Rachev ratio, generalized Rachev ratio), and other admissible performance measures are a strictly increasing function in the Sharpe ratio. The theoretical basis of this result is the location and scale property and two other plausible and mild conditions. Our result provides a decision-theoretic foundation for all these frequently used performance measures. Moreover, it might explain the empirical finding that all these measures typically lead to very similar rankings.

1. Introduction

The most popular reward-to-risk performance measure is the Sharpe ratio (see, e.g., Alexander and Baptista, 2010; Darolles and Gourieroux, 2010; Ding et al., 2009; Eling and Faust, 2010; Szakmary et al., 2010; Serban, 2010). The least restrictive sufficient condition for expected utility to imply Sharpe ratio rankings is the location and scale (LS) property (see Sinn, 1983; Meyer, 1987). This property requires that the random returns from the investment funds in the choice set differ from one another only by location and scale parameters. Schuhmacher and Eling (2011) argue that the LS property is also sufficient for expected utility to imply drawdown-based performance measure rankings. Hence, the same conditions that provide an expected utility foundation for the Sharpe ratio also provide a foundation for drawdown-based performance measures. Their result shows that drawdown-based performance measures will lead to the same ranking as the Sharpe ratio if the random returns satisfy the LS property.

Thus the question arises as to whether the LS property is sufficient to ensure consistency between expected utility and other performance measures that differ from the Sharpe ratio by the risk and reward measure employed. To answer this question, we argue that any admissible risk measure should satisfy two conditions: first, it should satisfy positive homogeneity, which is an important axiom in most axiomatic systems (see Kijima and Ohnishi, 1993; Pedersen and Satchell, 1998; Artzner et al., 1999; Rockafellar et al., 2006); second, adding a positive constant to an investment fund's random excess rate of return should not increase the investment fund's risk. This condition contains the mutually incompatible axioms of translation invariance (see Artzner et al., 1999) and shift invariance (see Kijima and Ohnishi, 1993; Pedersen and Satchell, 1998; Rockafellar et al., 2006) as special cases.

Similarly, any admissible reward measure should also satisfy two conditions: positive homogeneity and that adding a positive constant to an investment fund's random excess rate of return does increase the investment fund's reward. The main result is that under the LS property, any admissible performance measure is a strictly increasing function in the Sharpe ratio. An admissible performance measure is using admissible risk and reward measures.

This finding has two important implications. First, it provides a decision-theoretic foundation for lower-partial-moments, value-at-risk, and other admissible performance measures that differ from the Sharpe ratio by the risk and reward measure employed. Second, since the normal, the extreme value, and many other

* Corresponding author. Tel.: +49 341 9733670; fax: +49 341 9733679.
E-mail addresses: schuhmacher@wifa.uni-leipzig.de (F. Schuhmacher), martin.eling@unisg.ch (M. Eling).

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distributions commonly used in finance satisfy the LS property (see Schuhmacher and Eling, 2011), the finding may explain the empirical observation that rank correlations between the Sharpe ratio and alternative performance measures are extremely high (see Eling and Schuhmacher, 2007; Eling et al., 2010).

This paper is structured as follows. In Section 2 we present our main result. In Section 3 we demonstrate the main result using well-known risk and reward measures. A numerical illustration is presented in Section 4. We conclude in Section 5.

2. The main result

Let X_i ($i = 1, \ldots, n$) denote a random variable that represents investment fund i's excess rate of return (over the risk-free rate) at the end of a period, e.g., a month. The expected excess rate of return is $\mu(X_i)$, the standard deviation $\sigma(X_i)$, and the Sharpe ratio is defined for $\sigma(X_i) > 0$ as

$$S(X_i) = \mu(X_i)/\sigma(X_i). \tag{1}$$

Sinn (1983) and Meyer (1987) show that if the investment funds' returns are equal in distribution to one another except for location and scale (LS), a Sharpe ratio ranking can be derived based on expected utility. According to Feller (1966), Meyer (1987), and Levy (1989), a set of random variables X_i described by probability density functions $f_i(\cdot)$ satisfies the LS property if there exists some random variable Y with a probability density function $g(\cdot)$ such that $f_i(\cdot)$ differs from $g(\cdot)$ only by location parameter a_i and scale parameter $b_i > 0$. That is, there exist location and scale parameters a_i and $b_i > 0$ such that $b_i \cdot f_i(a_i + b_i \cdot y) = g(y)$, which means $X_i \sim a_i + b_i \cdot Y$, whereby "$\sim$" stands for "is equal in distribution to." Furthermore, the random variable Y can be standardized such that $\mu(Y) = 0$ and $\sigma(Y) = 1$. Hence, $\mu(X_i) = \mu(a_i + b_i \cdot Y) = a_i + b_i \cdot \mu(Y) = a_i$ and $\sigma(X_i) = \sigma(a_i + b_i \cdot Y) = b_i \cdot \sigma(Y) = b_i$. As a consequence, fund i's Sharpe ratio is $S(X_i) = a_i/b_i$.

We consider an alternative risk measure denoted by $\rho(X_i)$ and an alternative reward measure denoted by $\pi(X_i)$. The corresponding performance measure $P(X_i)$ is then defined for $\rho(X_i) > 0$ as

$$P(X_i) = \pi(X_i)/\rho(X_i). \tag{2}$$

In the literature, there are a number of important axiomatic systems for risk measures (see Kijima and Ohnishi, 1993; Pedersen and Satchell, 1998; Artzner et al., 1999; Rockafellar et al., 2006). One axiom common to all four axiomatic systems is positive homogeneity (PH), which requires

$$\rho(k \cdot X_i) = k \cdot \rho(X_i) \text{ for all } k > 0 \text{ and } \rho(0) = 0. \tag{3}$$

Positive homogeneity is Axiom R1 in Kijima and Ohnishi (1993), Axiom BP2 in Pedersen and Satchell (1998), Axiom PH in Artzner et al. (1999), and Axiom D2 in Rockafellar et al. (2006). Note that the first three papers require $\rho(k \cdot X_i) = k \cdot \rho(X_i)$ for all $k \geqslant 0$, while the last paper requires $\rho(k \cdot X_i) = k \cdot \rho(X_i)$ for all $k > 0$ and $\rho(0) = 0$. Both formulations are equivalent.

Positive homogeneity is the first condition every admissible risk measure needs to satisfy. According to Ortobelli et al. (2005) (see also Szegö, 2002; Frittelli and Gianin, 2002), positive homogeneity is a sufficient (but not a necessary) condition to add to convexity in order to obtain sublinearity (=coherence) à la Artzner et al. (1999). Note that positive homogeneity concerns the influence of multiplying the random variable by a constant. The second condition every admissible risk measure needs to satisfy concerns the influence of adding a constant to the random variable. According to Kijima and Ohnishi (1993), Pedersen and Satchell (1998), and Rockafellar et al. (2006), adding a constant should not change the risk measure (see Axiom R4 in Kijima and Ohnishi, 1993, Axiom BP4 in Pedersen and Satchell, 1998, and Axiom D1 in Rockafellar et al., 2006). Kijima and Ohnishi (1993) call this axiom "shift invariance":

$$\rho(X_i + s) = \rho(X_i). \tag{4}$$

According to Artzner et al. (1999), adding a constant should decrease the risk measure by the same amount. Artzner et al. (1999) call this axiom "translation invariance":

$$\rho(X_i + s) = \rho(X_i) - s. \tag{5}$$

Obviously, shift invariance and translation invariance are mutually incompatible. According to these (mutually incompatible) axioms, risk measures can be broadly categorized as deviation risk measures, in the sense of Rockafellar et al. (2006), when they satisfy shift invariance, or as coherent risk measures, in the sense of Artzner et al. (1999), when they satisfy translation invariance. *This leads to the second condition every admissible risk measure needs to satisfy, which is that adding a strictly positive constant to the random variable should not increase risk*. Following Ortobelli et al. (2005), we call this property functional translation invariance for a risk measure (FTIRisk)

$$\rho(X_i + s) \leqslant \rho(X_i) \text{ for } s > 0. \tag{6}$$

Obviously, this condition contains shift invariance and translation invariance as special cases. Coombs and Lehner (1981, p. 1116) call this a universal property of risk (see Sarin, 1987). According to Ortobelli et al. (2005), functional translation invariance is satisfied if for all real t and risky wealth W, the function $f(t) = \rho(X + t)$ is a continuous and non-increasing function.

The alternative reward measure should satisfy two conditions as well in order to be admissible. First, it should satisfy positive homogeneity, i.e.

$$\pi(k \cdot X_i) = k \cdot \pi(X_i). \tag{7}$$

Second, adding a strictly positive constant to the random variable should increase the reward measure. For simplicity, we call this property functional translation invariance for reward measures (FTIReward).

$$\pi(X_i + s) > \pi(X_i) \text{ for } s > 0. \tag{8}$$

Note the two asymmetries between FTIRisk and FTIReward. The first asymmetry is that FTIRisk requires a *decrease*, whereas FTIReward requires an *increase* of the measure. The second asymmetry is that FTIRisk requires a *weak* inequality, while FTIReward requires a *strict* inequality. This asymmetry stems from the fact that the risk measure might be a deviation measure, which means that adding a constant has no effect on the risk measure. The reward measure is never a deviation measure.

Before stating our main result, we need the following proposition, which describes under the LS property the consequence of positive homogenous risk and reward measures for the performance measure.

Proposition 1. *Assume two funds with the LS property and identical Sharpe ratios, $S(X_1) = S(X_2)$. If both the alternative risk measure $\rho(X_i)$ and the reward measure $\pi(X_i)$ satisfy positive homogeneity, then the corresponding performance measures for the two funds are also identical, $P(X_1) = P(X_2)$.*

Proof. See Appendix. □

Proposition 1 is important because it implies that a positive homogenous performance measure is a function of the Sharpe ratio. The central question, which will be answered next, is under what conditions is it an increasing function of the Sharpe ratio?