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Steady-state mean squared error and tracking performance analysis of the quasi-OBE algorithm

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ABSTRACT

The quasi-OBE (QOBE) algorithm is a set-membership adaptive filtering algorithm based on the principles of optimal bounding ellipsoid (OBE) processing. This algorithm can provide enhanced convergence and tracking performance as well as reduced average computational complexity in comparison with the more traditional adaptive filtering algorithms such as the recursive least squares (RLS) algorithm. In this paper, we analyze the steady-state mean squared error (MSE) and tracking performance of the QOBE algorithm. For this purpose, we derive energy conservation relation of the QOBE algorithm. The analysis leads to a nonlinear equation whose solution gives the steady-state MSE of the QOBE algorithm in both stationary and nonstationary environments. We prove that there is always a unique solution for this equation. The results predicted by the analysis show good agreement with the simulation experiments.

1. Introduction

Set-membership adaptive filtering (SMAF) algorithms are known for their superiority over the more classical estimation techniques in three aspects. First, they provide sets of acceptable estimates rather than single estimates. Second, they feature improved convergence and tracking properties generally due to the execution of some kind of set-theoretic optimization, e.g., optimization of the step size or the weighting sequence. Third, they enjoy a data-discerning update strategy, which enables them to check for the innovation in the new data at each time instant and determine whether an update is required or not. In other words, they perform an update only when it can improve the quality of estimation and obviate the expense of updating when there is no useful information in the incoming data. As a result, in addition to the enhanced convergence and tracking performance, they can also provide an appreciable reduction in the average computational complexity [1–3].

The optimal bounding ellipsoids (OBE) algorithms are well-established SMAF algorithms that tightly outer-bound the set of feasible solutions in the associated parameter space using ellipsoids. They optimize the size of the ellipsoids in some meaningful sense. Different optimality criteria have led to different OBE algorithms. Among them, the quasi-OBE (QOBE) algorithm[1] [4,5] is particularly interesting since it shares many of the desired features of the various OBE algorithms. Furthermore, it incorporates simple but efficient innovation check and optimal weight calculation processes, which make it computationally more efficient than other OBE algorithms.

The SMAF algorithms generally have high degrees of nonlinearities, which make their performance analysis complicated, especially at the transient state. The QOBE algorithm involves both data nonlinearity [6] and error nonlinearity [7]. Few works have been reported on the

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[1] This algorithm was originally published as the *Bounding Ellipsoidal Adaptive CONstrained least-squares* (BEACON) algorithm [5].

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theoretical performance analysis of the SMAF algorithms. Steady-state mean squared error (MSE) of the set-membership normalized least mean squares (SM-NLMS) algorithm [8] is analysed in [9,10] and steady-state MSE of the set-membership affine projection (SM-AP) algorithm [11] is analyzed in [12–14]. Analytical results on the steady-state MSE performance of the set-membership binormalized data-reusing LMS algorithm and the multichannel filtered-x set-membership affine projection algorithm are given in [15] and [16], respectively. The work in [4] is also dedicated to prove that, with persistent excitation [17], the QOBE algorithm exhibits central-estimator convergence while its associated hyperellipsoidal membership set cannot converge to a point set. To the best of our knowledge, no steady-state MSE or tracking performance analysis of the QOBE algorithm has been reported in the literature to date.

In this paper, we present a steady-state MSE and tracking performance analysis of the QOBE algorithm. The classic approach is to analyze the transient behavior of the adaptive filter and then obtain the steady-state MSE as a limiting behavior of the transient MSE [18,19]. However, due to high nonlinearity of the QOBE algorithm, this approach would be very difficult. To circumvent the analysis of the transient behavior and its complications, we employ the energy conservation argument [20,21] and initiate our steady-state analysis from the energy conservation relation of the QOBE algorithm. For the tracking performance analysis, we consider a random walk model for time variations of the unknown system in a nonstationary environment. The outcome of the analysis is a nonlinear equation whose solution yields the theoretical steady-state MSE of the QOBE algorithm in both stationary and nonstationary environments. Simulations confirm the accuracy of the theoretical predictions for the steady-state MSE in addition to justifying the assumptions made in the derivations.

In Section 2, we describe the QOBE algorithm. We analyze its steady-state MSE and tracking performance in Sections 3 and 4, provide some simulation results in Section 5, and conclude the paper in Section 6.

2. The QOBE algorithm

Let us consider the affine-in-parameter model:

$$d_n = \boldsymbol{\omega}^T \mathbf{x}_n + \eta_n \tag{1}$$

where $d_n \in \mathbb{R}$ is the reference signal at time index $n \in \mathbb{N}$, $\boldsymbol{\omega} \in \mathbb{R}^L$ is the column vector of the unknown system parameters, $\mathbf{x}_n \in \mathbb{R}^L$ is the input vector, $\eta_n \in \mathbb{R}$ accounts for background noise, and superscript T denotes matrix/vector transposition.

Conventional filtering schemes produce an estimate of $\boldsymbol{\omega}$ by minimizing a cost function, which is usually a direct function of the estimation error. On the contrary, set-membership filtering algorithms estimate a set of feasible solutions rather than a single-point estimate by specifying a bound on the magnitude of the estimation error over the model space of interest. The model space, $\mathcal{S}$, comprises all the input-reference signal pairs used in the estimation. Hence, any parameter vector, $\mathbf{w}$, resulting in an error less than a predetermined bound, $\gamma > 0$, for all data pairs from $\mathcal{S}$ is an acceptable solution. The *feasibility set* contains all these possible solutions and is defined as

$$\Theta = \bigcap_{(\mathbf{x},d)\in\mathcal{S}} \{\mathbf{w} \in \mathbb{R}^L : |d - \mathbf{w}^T\mathbf{x}| \le \gamma\}.$$

Direct calculation of Θ is often unachievable since, in practice, $\mathcal{S}$ is not fully known and may be time-varying. Therefore, SMAF algorithms aim to estimate a minimal set estimate of Θ at each time instant. This set is called the *membership set* and is defined as

$$\Psi_n = \bigcap_{i=1}^{n} \Phi_i$$

where Φ_n is the observation-induced *constraint set* at time instant n:

$$\Phi_n = \{\mathbf{w} \in \mathbb{R}^L : |d_n - \mathbf{w}^T\mathbf{x}_n| \le \gamma\}.$$

Obviously, $\Theta \subset \Psi_n$. However, Ψ_n is a convex polytope in $\mathcal{S}$ and usually too complex to compute. One solution is to outer-bound Ψ_n at each iteration by a mathematically tractable ellipsoid E_n. This is the basic idea behind the OBE algorithms. Given an initial ellipsoid

$$\mathrm{E}_0 = \{\mathbf{w} \in \mathbb{R}^L : (\mathbf{w} - \mathbf{w}_0)^T \mathbf{P}_0^{-1} (\mathbf{w} - \mathbf{w}_0) \le \sigma_0\}$$

with some suitably chosen initial estimates $\mathbf{w}_0 \in \mathbb{R}^L$, $\mathbf{P}_0^{-1} \in \mathbb{R}^{L\times L}$, and $\sigma_0 > 0$, an OBE algorithm recursively computes E_n using the knowledge of E_{n-1} in such a way that

$$\mathrm{E}_n \supset (\Phi_n \cap \mathrm{E}_{n-1}) \supset \Psi_n \forall n. \tag{2}$$

As E_n outer-bounds the intersection of Φ_n and E_{n-1}, Θ lies within E_n. To satisfy (2), E_n is calculated via

$$\mathrm{E}_n = \{\mathbf{w} \in \mathbb{R}^L : (\mathbf{w} - \mathbf{w}_n)^T \mathbf{P}_n^{-1} (\mathbf{w} - \mathbf{w}_n) \le \sigma_n\}$$

where

$$\mathbf{P}_n^{-1} = \mathbf{P}_{n-1}^{-1} + \lambda_n \mathbf{x}_n \mathbf{x}_n^T, \tag{3}$$

$$e_n = d_n - \mathbf{w}_{n-1}^T \mathbf{x}_n, \tag{4}$$

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \lambda_n \mathbf{P}_n \mathbf{x}_n e_n, \tag{5}$$

$$\sigma_n = \sigma_{n-1} - \frac{\lambda_n e_n^2}{1 + \lambda_n G_n} + \lambda_n \gamma^2,$$

and

$$G_n = \mathbf{x}_n^T \mathbf{P}_{n-1} \mathbf{x}_n.$$

The centroid of E_n, $\mathbf{w}_n$, can be regarded as a point estimate at time instant n and e_n is the *a priori* estimation error. The QOBE (also known as BEACON) algorithm calculates the optimal weighting factor, λ_n, by minimizing σ_n under the constraint that $\lambda_n > 0$. The result is a simple but efficient update-checking rule and a relatively simple expression for λ_n:

$$\lambda_n = \begin{cases} \frac{1}{G_n}\left(\frac{|e_n|}{\gamma} - 1\right) & \text{if } |e_n| > \gamma \\ 0 & \text{if } |e_n| \le \gamma \end{cases}. \tag{6}$$

In practice, $\mathbf{P}_n$ is updated rather than $\mathbf{P}_n^{-1}$ by applying the matrix inversion lemma [22] to (3):

$$\mathbf{P}_n = \mathbf{P}_{n-1} - \frac{\lambda_n \mathbf{P}_{n-1} \mathbf{x}_n \mathbf{x}_n^T \mathbf{P}_{n-1}}{1 + \lambda_n G_n}. \tag{7}$$