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ScienceDirect
Journal of Sound and Vibration 311 (2008) 56–73
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Comparisons between harmonic balance and nonlinear output frequency response function in nonlinear system analysis

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Received 25 January 2007; received in revised form 22 August 2007; accepted 22 August 2007
Available online 22 October 2007

Abstract

By using the Duffing oscillator as a case study, this paper shows that the harmonic components in the nonlinear system response to a sinusoidal input calculated using the nonlinear output frequency response functions (NOFRFs) are one of the solutions obtained using the harmonic balance method (HBM). A comparison of the performances of the two methods shows that the HBM can capture the well-known *jump phenomenon*, but is restricted by computational limits for some strongly nonlinear systems and can fail to provide accurate predictions for some harmonic components. Although the NOFRFs cannot capture the *jump phenomenon*, the method has few computational restrictions. For the nonlinear damping systems, the NOFRFs can give better predictions for all the harmonic components in the system response than the HBM even when the damping system is strongly nonlinear.

1. Introduction

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. Various approaches, including the perturbation method [1–6], multiple scale method [7–12], and the harmonic balance method (HBM) [12–23] have been developed to study the forced periodic motions of these nonlinear oscillators. Among these methods, the HBM is considered to be one of powerful methods capable of handling strongly nonlinear behaviours and, it can converge to an accurate periodic solution for smooth nonlinear systems [13].

The HBM method is based on the assumption that the system time domain response can be expressed in the form of a Fourier series. Therefore, the HBM is usually used to study nonlinear systems where the output responses of which are periodic in time. Such nonlinear systems range from models as simple as the Duffing oscillator [14] to more complex models such as cracked rotors [15]. More applications of the HBM can be found in the study of the nonlinear response of airfoils [16–17], nonlinear conservative systems [18], hysteretic

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.08.035

two-degree-of-freedom systems [19], the third-order (jerk) differential equations [20] and the Jeffcott rotor [21]. By using the HBM, some interesting phenomena unique to nonlinear systems have been observed, among which the most well known is *jump phenomenon* where the response amplitude of a nonlinear oscillator changes suddenly at some critical value of the frequency of the excitation [13]. Although the basic idea of the HBM is quite simple (to substitute a Fourier series form solution of the system time domain response into the governing equations of the system under study, and to equate coefficients of the same harmonic components), its implementation is actually not easy [14]. First, if many frequency components are taken into account in the HBM to reach accurate results, it is highly possible for the HBM to fail. Second, for the Duffing oscillator, the HBM is typically easy to implement but, for models with more complex nonlinearities, it may be very difficult or impossible to implement. Moreover, it is always necessary to write specific computation programs for different nonlinear models [14], and that is why improved HBM need to be developed.

The Volterra series approach [22–24] is another powerful method for the analysis of nonlinear systems, which extends the well-known concept of the convolution integral for linear systems to a series of multidimensional convolution integrals. The Fourier transforms of the Volterra kernels, called generalised frequency response functions (GFRFs) [25], are an extension of the linear frequency response function (FRF) to the nonlinear case. If a differential equation or discrete-time model is available for a nonlinear system, the GFRFs can be determined using the algorithm in Refs. [26–28]. However, the GFRFs are multidimensional functions [29,30], which can be much more complicated than the linear FRF and can be difficult to measure, display and interpret in practice. Recently, a novel concept known as nonlinear output frequency response functions (NOFRFs) was proposed by the authors [31]. The concept can be considered to be an alternative extension of the classical FRF for linear systems to the nonlinear case. NOFRFs are one-dimensional functions of frequency, which allows the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems and provides great insight into the mechanisms which dominate many nonlinear behaviours. For a nonlinear system subjected to a harmonic input, the response could also be described by a Fourier series using the NOFRFs. The present study is concerned with a comparison study between the NOFRFs and HBM methods in the analysis of a class of nonlinear systems.

2. Harmonic balance method (HBM)

In the HBMs [14], the solution of a nonlinear system is assumed to be of the form of a truncated Fourier series:

$$y(t) = d_0 + \sum_{n=1}^{\overline{N}} (a_n \cos(n\omega t) + b_n \sin(n\omega t)), \tag{1}$$

where d_0, a_n and b_n $(n = 1, \ldots, \bar{N})$ are known as the HB solution Fourier coefficients, and $\bar{N}$ the number of harmonic components used in the HB-truncated Fourier series expansion. The principle of the HBM can be illustrated using the Duffing oscillator:

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 y^3 = A \cos(\omega t), \tag{2}$$

where m, c, k_1 and k_3 are the parameters of the mass, damping and stiffness of the system respectively. A and ω are the external excitation force amplitude and frequency of the oscillator.

The Fourier expansions of the first and second derivatives of the output of system (2) are:

$$\dot{y}(t) = \sum_{n=1}^{\bar{N}} n\omega(-a_n \sin(n\omega t) + b_n \cos(n\omega t)), \tag{3}$$

$$\ddot{y}(t) = \sum_{n=1}^{\bar{N}} -n^2\omega^2(a_n \cos(n\omega t) + b_n \sin(n\omega t)). \tag{4}$$