Nonlinear Analysis: Hybrid Systems 7 (2013) 54–67

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Binary input reconstruction for linear systems: A performance analysis

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ARTICLE INFO

Keywords:
Linear systems
Binary input reconstruction
Online decoding
Markov processes
Iterated random functions

ABSTRACT

Recovering the digital input of a time-discrete linear system from its (noisy) output is a significant challenge in the fields of data transmission, deconvolution, channel equalization, and inverse modeling. A variety of algorithms have been developed for this purpose in the last decades, addressed to different models and performance/complexity requirements. In this paper, we implement a straightforward algorithm to reconstruct the binary input of a one-dimensional linear system with known probabilistic properties. Although suboptimal, this algorithm presents two main advantages: it works online (given the current output measurement, it decodes the current input bit) and has very low complexity. Moreover, we can theoretically analyze its performance: using results on convergence of probability measures, Markov processes, and Iterated Random Functions we evaluate its long-time behavior in terms of mean square error.

1. Introduction

Consider the input/output linear system

$$\begin{cases} x_k = qx_{k-1} + wu_{k-1} & k = 1, \dots, K \\ y_k = cx_k + n_k \end{cases} \tag{1}$$

with $K \in \mathbb{N}$ (possibly tending to infinity), $u_k \in \{0, 1\}$ for $k = 0, \dots, K-1$, $x_k \in \mathbb{R}$ for $k = 0, \dots, K$, $y_k, n_k \in \mathbb{R}$ for $k = 1, \dots, K$, $q, w, c \in \mathbb{R}$, and $q \in (0, 1)$ to preserve stability. Our aim is to recover the binary input u_k, in an online fashion, given the output y_k corrupted by a noise n_k. To this purpose, we retrieve a low-complexity algorithm introduced in [1] and discussed in [2,3], and we propose a comprehensive theoretical analysis of its performance. As a result of the analysis, we will be able to evaluate the performance as a function of the system's parameters.

The digital signal reconstruction problem is a paradigm in data transmissions, where signals arising from finite alphabets are sent over noisy continuous channels, and in hybrid frameworks, where digital and analog signals have to be merged in the same system. In [1], a slightly different instance of model (1) was derived as time discretization of a convolution system and the input estimation described as a deconvolution problem. The same can be achieved for model (1): if we consider the system

$$\begin{cases} x'(t) = ax(t) + bu(t) & t \in [0, T] \\ y(t) = cx(t) + n(t) & x(0) = x_0 \\ u(t), x(t), y(t), & a, b, c \in \mathbb{R}, a < 0 \end{cases} \tag{2}$$

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1751-570X/$ – see front matter
doi:10.1016/j.nahs.2012.07.007

we have

$$x(t) = e^{ta}x_0 + b\int_0^t e^{a(t-s)}u(s)ds. \tag{3}$$

Given

$$u(t) = \sum_{k=0}^{K-1} u_k \mathbb{1}_{[k\tau,(k+1)\tau[}(t), \quad u_k \in \mathcal{U} = \{0, 1\}, \tau > 0 \tag{4}$$

we can discretize in the following way: by defining

$$\begin{aligned} x_k &:= x(k\tau) \quad \text{for } k = 0, 1, \ldots, K = T/\tau \\ q &:= e^{\tau a} \in (0, 1) \\ w &:= b\frac{1 - e^{\tau a}}{-a} = -\frac{b}{a}(1 - q) \end{aligned} \tag{5}$$

we obtain

$$\begin{aligned} x_k &= q^k x_0 + bq^k \int_0^{k\tau} e^{-as} \sum_{h=0}^{K-1} u_h \mathbb{1}_{[h\tau,(h+1)\tau[}(s)ds \\ &= q^k x_0 + bq^k \sum_{h=0}^{k-1} u_h \int_{h\tau}^{(h+1)\tau} e^{-as}ds \\ &= q^k x_0 + \frac{b}{-a}q^k \sum_{h=0}^{k-1} u_h e^{-a(h+1)\tau}(1 - e^{a\tau}) \\ &= q^k x_0 + w\sum_{h=0}^{k-1} u_h q^{k-1-h} \end{aligned} \tag{6}$$

from which we have the recursive formula

$$x_k = qx_{k-1} + wu_{k-1}. \tag{7}$$

In system (2), recovering $u(t)$ basically consists in the inversion of the convolution integral

$$y(t) = ce^{ta}x_0 + cb\int_0^t e^{a(t-s)}u(s)ds + n(t)$$

(where $n(t)$ represents an additive noise), which is a long-standing mathematical ill-posed problem: small observation errors may produce defective solutions. Several estimation approaches have been studied in the last fifty years and the literature on deconvolution is widespread: we refer the reader to early papers [4,5] and to later ones [6–9], which also show some possible applications in geophysics, astronomy, image processing and biomedical systems. For more references, see [10].

Most of the known deconvolution methods exploit the regularity of the input function to provide good estimations. This work instead is a contribution for deconvolution in case of discontinuous input functions.

Considering a binary alphabet, which has been chosen mainly to keep the analysis straightforward, is consistent with many applications: the output of several digital devices, such as computers and detection devices [2], are binary. Nevertheless, the algorithm and the analysis presented in this paper could be generalized to larger alphabets with not much effort.

In [1], low-complexity decoding algorithms were introduced, derived from the optimal BCJR [11] algorithm, and applied to perform the deconvolution of the system (2) with $a = 0$ and $b = c = 1$. In this work, we apply the simplest of those algorithms, the so-called One State Algorithm (OSA for short) to the system (1). We then describe the performance in terms of Mean Square Error (MSE) for long-time transmissions, through a probabilistic analysis arising from the Markovian behavior of the algorithm. The scheme of the analysis is the same proposed in [1], but leads to completely different scenarios: while for $a = 0$ and $b = c = 1$ standard ergodic theorems for denumerable Markov Processes were sufficient to compute the MSE, in the present case the denumerable model does not proved the expected results, and more sophisticated arguments are used, arising from Markov Processes, Iterated Random Functions (IRF for short) and sequences of probability measures.

The paper is organized as follows. In Section 2 we complete the description of the system, giving some observations and probabilistic assumptions; in Sections 3 and 4, we present our algorithm and some simulations. The core of the paper is the performance analysis provided in Section 5. Finally we propose some concluding observations. Notice that Sections 2 and 3 mainly retrieve the model presented in [1,2].