Systems & Control Letters 63 (2014) 50–56

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

# LMI approach to linear positive system analysis and synthesis

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ARTICLE INFO

*Article history:*
Received 29 August 2012
Received in revised form
20 September 2013
Accepted 1 November 2013
Available online 1 December 2013

*Keywords:*
Positive system
Diagonal Lyapunov matrix
LMI
Duality

ABSTRACT

This paper is concerned with the analysis and synthesis of linear positive systems based on linear matrix inequalities (LMIs). We first show that the celebrated Perron–Frobenius theorem can be proved concisely by a duality-based argument. Again by duality, we next clarify a necessary and sufficient condition under which a Hurwitz stable Metzler matrix admits a diagonal Lyapunov matrix with some identical diagonal entries as the solution of the Lyapunov inequality. This new result leads to an alternative proof of the recent result by Tanaka and Langbort on the existence of a diagonal Lyapunov matrix for the LMI characterizing the $H_\infty$ performance of continuous-time positive systems. In addition, we further derive a new LMI for the $H_\infty$ performance analysis where the variable corresponding to the Lyapunov matrix is allowed to be non-symmetric. We readily extend these results to discrete-time positive systems and derive new LMIs for the $H_\infty$ performance analysis and synthesis. We finally illustrate their effectiveness by numerical examples on robust state-feedback $H_\infty$ controller synthesis for discrete-time positive systems affected by parametric uncertainties.



## 1. Introduction

This paper is concerned with the analysis and synthesis of linear time-invariant (LTI) positive systems. A linear system is said to be positive (or more accurately, internally positive) if its state and output are both nonnegative for any nonnegative initial state and nonnegative input. Because of this strong property, there are remarkable, and very peculiar results that are valid only for positive systems. Among them, the existence of a diagonal Lyapunov matrix that characterizes stability is well known [1,2].

Recently, Shorten et al. showed that the peculiar "diagonal stability result" can be proved by means of the duality theory in convex optimization. They further obtained new results on the stability of switched positive systems [3–6]. Along this line, Tanaka and Langbort proved that the KYP-type linear matrix inequality (LMI) characterizing the $H_\infty$ performance of positive systems admits a diagonal Lyapunov matrix [7]. These recent results indicate that the duality theory is a powerful tool for positive system analysis.

Along the same line, in this paper, we develop duality-based arguments for positive system analysis. Our novel contribution can be summarized as follows:

1. We provide a duality-based concise proof of the Perron–Frobenius theorem [1,2]. In addition to the existence of the Frobenius eigenvalue, we show the existence of the nonnegative eigenvector by duality.
2. Again by a duality-based argument, we clarify a necessary and sufficient condition under which a Hurwitz stable Metzler matrix admits a diagonal Lyapunov matrix with some identical diagonal entries. This condition leads to an alternative proof of the result in [7]. The analysis is partly motivated from the observation that the $L_2$ and $L_1$ induced norm analysis of positive systems can be transformed into the stability analysis of appropriately constructed positive systems [8,9].
3. We derive new LMI conditions for the stability and $H_\infty$ performance analysis of continuous-time positive systems, where the common positive definiteness constraint on the Lyapunov matrix $P$ as in $P \succ 0$ can be relaxed to $P + P^T \succ 0$. This implies that $P$ is not necessarily required to be symmetric.
4. We extend the above results to discrete-time positive systems and derive new LMIs for the $H_\infty$ performance analysis and synthesis, some of which are reported also in [10]. We illustrate the effectiveness of these new LMIs by numerical examples on structured robust state-feedback $H_\infty$ controller synthesis for discrete-time positive systems affected by parametric uncertainties. Even though we provide LMI-based formulations in this paper, it is known that linear-programming-based formulation is possible in the case where the plant is SISO, see, ex., [8,11].

Note that a conference version of this paper was presented in [12]. In the current paper we include new LMI results for discrete-time positive systems. In particular, we show that a given discrete-time positive system can be converted into a continuous-time

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http://dx.doi.org/10.1016/j.sysconle.2013.11.001

positive system preserving the stability and the $H_\infty$ norm. This enables us to derive new LMIs for the $H_\infty$ performance analysis and synthesis of discrete-time positive systems.

We use the following notations in this paper. First, we denote by $\mathbb{S}^n_{++}$ ($\mathbb{S}^n_{+}$) the set of positive (semi)definite matrices of size $n$. For a symmetric matrix $X \in \mathbb{R}^{n\times n}$, we also write $X \succ 0$ ($X \succeq 0$) to denote that $X$ is positive (semi)definite. Similarly, we write $X \prec 0$ ($X \preceq 0$) to denote that $X$ is negative (semi)definite. In addition, we denote by $\mathbb{D}^n_{++}$ the set of diagonal, and positive definite matrices of size $n$. For $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}\{A\} = A + A^T$. The notation $\lambda(A)$ stands for the set of the eigenvalues of $A$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Hurwitz* stable if $\max_{\lambda\in\lambda(A)} \mathrm{Re}\,\lambda < 0$, and is said to be *Schur* stable if $\max_{\lambda\in\lambda(A)} |\lambda| < 1$. For two given matrices $A$ and $B$ of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all $(i, j)$, where $A_{ij}$ ($B_{ij}$) stands for the $(i, j)$-entry of $A(B)$. We also define

$$\mathbb{R}^{n\times m}_{++} := \left\{A \in \mathbb{R}^{n\times m},\ A > 0\right\}, \qquad \mathbb{R}^{n\times m}_{+} := \left\{A \in \mathbb{R}^{n\times m},\ A \geq 0\right\}.$$

Finally, for a given $A \in \mathbb{R}^{n\times n}$, we define by $\mathcal{D}(A) \in \mathbb{R}^n$ the vector composed of the diagonal entries, i.e., $\mathcal{D}(A) := [A_{11} \ \cdots \ A_{nn}]^T$.

## 2. Fundamentals of positive systems

In this brief section, we gather basic definitions and fundamental results for positive system analysis. See [1,2] for a more complete treatment.

**Definition 1** (*Positive Linear System [1]*)**.** A linear system is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

A system satisfying the condition in Definition 1 is often called *internally* positive, to make a clear distinction from *externally* positive systems. Since we only deal with internally positive systems in this paper, we simply denote them by positive as in Definition 1.

**Definition 2** (*Metzler Matrix [1]*)**.** A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ ($i \neq j$).

**Proposition 1** (*[1]*)**.** *Let us consider the continuous-time LTI system described by*

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \\ z(t) = Cx(t) + Dw(t). \end{cases} \tag{1}$$

*Then, this system is positive if and only if $A$ is Metzler, $B \geq 0$, $C \geq 0$, and $D \geq 0$.*

**Proposition 2** (*[1]*)**.** *Let us consider the discrete-time LTI system described by*

$$G_{\mathrm{d}} : \begin{cases} x(k+1) = A_{\mathrm{d}}x(k) + B_{\mathrm{d}}w(k), \\ \quad\ z(k) = C_{\mathrm{d}}x(k) + D_{\mathrm{d}}w(k). \end{cases} \tag{2}$$

*Then, this system is positive if and only if $A_{\mathrm{d}} \geq 0$, $B_{\mathrm{d}} \geq 0$, $C_{\mathrm{d}} \geq 0$, and $D_{\mathrm{d}} \geq 0$.*

In the following, we denote by $\mathbb{M}^n$ the set of the Metzler matrices of size $n$. The next theorem summarizes basic results for the Hurwitz stability of Metzler matrices.

**Proposition 3** (*[1,2]*)**.** *For a given $A \in \mathbb{M}^n$, the following conditions are equivalent.*

(i) *The matrix $A$ is Hurwitz stable.*
(ii) *For any $h \in \mathbb{R}^n_+ \setminus \{0\}$, the row vector $h^TA$ has at least one strictly negative entry.*
(iii) *There exists $h \in \mathbb{R}^n_{++}$ such that $h^TA < 0$.*
(iv) *There exists $g \in \mathbb{R}^n_{++}$ such that $Ag < 0$.*
(v) *The matrix $A$ is nonsingular and satisfies $A^{-1} \leq 0$.*

## 3. Preliminary results

In this section, we introduce preliminary results that are effective for positive system analysis. For conciseness, with a slight abuse of notation, we first make the following definition.

**Definition 3.** For a given $H \in \mathbb{S}^n_+$, we define $\bar{h} \in \mathbb{R}^n_+$ by $\bar{h}_i = \sqrt{H_{ii}}$ ($i = 1, \ldots, n$).

Under this definition, the following three lemmas hold.

**Lemma 1.** *For a given $H \in \mathbb{S}^n_+$, we have*

$$(\bar{h}\bar{h}^T)_{ii} = H_{ii}, \qquad (\bar{h}\bar{h}^T)_{ij} \geq H_{ij} \quad (i \neq j). \tag{3}$$

**Proof.** The first equality is obvious. On the other hand, since $H \succeq 0$, we have $H_{ii}H_{jj} \geq H_{ij}^2$ for $i \neq j$. It follows that $\sqrt{H_{ii}}\sqrt{H_{jj}} \geq H_{ij}$. Therefore, on the $(i, j)$ entry of $\bar{h}\bar{h}^T - H$, we have $(\bar{h}\bar{h}^T)_{ij} - H_{ij} = \sqrt{H_{ii}}\sqrt{H_{jj}} - H_{ij} \geq 0$. This completes the proof. ■

**Lemma 2.** *For given $A \in \mathbb{M}^n$ and $H \in \mathbb{S}^n_+$, we have $\mathcal{D}(\bar{h}\bar{h}^TA) \geq \mathcal{D}(HA)$.*

**Proof of Lemma 2.** Since $A \in \mathbb{M}^n$ and hence $A_{ij} \geq 0$ ($i \neq j$), we see from Lemma 1 that

$$\begin{aligned}(\bar{h}\bar{h}^TA)_{ii} &= (\bar{h}\bar{h}^T)_{ii}A_{ii} + \sum_{j=1, j\neq i}^{n} (\bar{h}\bar{h}^T)_{ij}A_{ji} \\ &\geq H_{ii}A_{ii} + \sum_{j=1, j\neq i}^{n} H_{ij}A_{ji} \\ &= (HA)_{ii}.\end{aligned}$$

This completes the proof. ■

This lemma in particular implies that if there exists $H \in \mathbb{S}^n_+$ that satisfies $\mathcal{D}(HA) \geq 0$ for a given $A \in \mathbb{M}^n$, then exactly the same property $\mathcal{D}(\bar{h}\bar{h}^TA) \geq 0$ holds with the rank-one matrix $\bar{h}\bar{h}^T$.

**Lemma 3.** *For given $A \in \mathbb{M}^n$ and $h_1, h_2 \in \mathbb{R}^n_+$, the following conditions are equivalent.*

(i) $\mathcal{D}\left(h_1(h_1^TA + h_2^T)\right) \geq 0$.
(ii) $h_1^TA + h_2^T \geq 0$.

**Proof of Lemma 3.** Since (ii) ⇒ (i) is obvious, we prove (i) ⇒ (ii) by contradiction. To this end, suppose $(h_1^TA + h_2^T)_i < 0$. Then, since $A$ is Metzler and $h_1, h_2 \in \mathbb{R}^n_+$, we have $A_{ii} < 0$ and $h_{1,i} > 0$. Hence $h_{1,i}(h_1^TA + h_2^T)_i < 0$, which clearly contradicts (i). ■

## 4. Duality-based proofs for Perron–Frobenius theorem

The next theorem is widely known as the Perron–Frobenius Theorem. It states that, among all the eigenvalues of a nonnegative matrix, the one with the largest modulus is located on the right-hand side of the real axis.

**Theorem 1** (*Perron–Frobenius Theorem [1,2]*)**.** *Suppose $A \in \mathbb{R}^{n\times n}_+$ is given. Then, $A$ has a nonnegative eigenvalue $\alpha$ such that $\alpha = \max_{\lambda\in\lambda(A)} |\lambda|$. Moreover, the eigenvector $g$ corresponding to the eigenvalue $\alpha$ satisfies $g \geq 0$.*[1]

This theorem is undoubtedly the central result in positive system analysis. It has a vast range of application areas such as biology, sociology and stochastic system analysis (see, ex., [13] and references cited therein). This theorem is proved, for example in [2], by

[1] Under the assumption that $A$ is irreducible, the Perron–Frobenius Theorem ensures $g > 0$ that is stronger than $g \geq 0$. See [1,2] for details.