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Cone Conditions in General Equilibrium Theory[1]

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The modern convex-analytic rendition of the classical welfare theorems characterizes optimal allocations in terms of supporting properties of preferences by non-zero prices. While supporting convex sets in economies with finite dimensional commodity spaces is usually a straightforward application of the separation theorem, it is not that automatic in economies with infinite dimensional commodity spaces. In the last 30 years several characterizations of the supporting properties of convex sets by non-zero prices have been obtained by means of cone conditions. In this paper, we present a variety of cone conditions, study their interrelationships, and illustrate them with many examples. *Journal of Economic Literature* Classification Numbers: D46, D51. © 2000 Academic Press

1. INTRODUCTION

Of the many insights of the old neoclassical school of economics, the characterization of economic optimality in terms of the equality of marginal rates of substitution, has remained a most enduring (and endearing) result in economic theory. The modern convex-analytic version of this

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classical insight characterizes optimality in terms of the supporting of preferences by a continuous linear functional (a price). Such a characterization is possible in the presence of finitely many commodities since every closed convex set in a finite dimensional space can be supported by a non-zero linear functional at each of its boundary points. This remarkable property can be attributed to a strictly finite dimensional peculiarity which ensures that every finite dimensional convex set has a non-empty interior in the smallest affine subspace that contains it; see for instance [17, Theorem 11.2.7, p. 341].

However, this supporting property fails when there are infinitely many commodities and when the arena of discourse is an infinite dimensional commodity space. In such a setting, supporting optimal allocations by means of prices (the second welfare theorem) is far more onerous a problem. In fact, convex sets with empty interior arise naturally in many economic models with infinitely many commodities. For example, if the positive cone of the commodity space has an empty interior, then every lower bounded consumption set has an empty interior.

This difficulty has been a subject of investigation throughout the second half of the twentieth century. Infinite dimensional results that are related to the second welfare theorem appeared quite early in the literature; see the works of G. Debreu [24], E. Malinvaud [35], M. Majumdar [34], B. Peleg and M. E. Yaari [38], and R. Radner [41]. Furthermore, since the works of K. J. Arrow [15], G. Debreu [24], and T. F. Bewley [18], it has become apparent that one of the major differences between economic models with finite and infinite dimensional commodity spaces is that in the finite dimensional setting the positive cone of the commodity space has an interior point.[3] Therefore, the standard infinite dimensional setting is one where the well-known cheeper point problem cannot be readily assumed away—and one may even appreciate the problem elucidated here by considering curious finite dimensional examples that "mimic" this infinite dimensional difficulty.[4]

A solution to this problem was presented in a seminal paper by A. Mas-Colell [36]. His solution was based on the works by Aliprantis and

[3] Recall that the positive cone of an infinite dimensional commodity space has a non-empty interior basically only if it is a majorizing subspace of some $C(\Omega)$-space (see [2, Sect. 7.5])—which is not the most appropriate setting for many economic models.

[4] For example, suppose that there are three commodities, i.e., the commodity space is $\mathbb{R}^3$. Assume that consumption sets are the positive orthant of $\mathbb{R}^3$ and suppose that the price space is smaller than $\mathbb{R}^3$ and is in fact some *two* dimensional subspace E of $\mathbb{R}^3$. It is clear that if we require that all decentralizing prices be elements of E instead of $\mathbb{R}^3$, then one needs extra assumptions on preferences to guarantee the validity of this stronger version of the second welfare theorem. The problem here is that the positive orthant of $\mathbb{R}^3$ has no interior points in the weak topology $\sigma(\mathbb{R}^3, E)$.