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Arc reversals in hybrid Bayesian networks with deterministic variables

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ABSTRACT

This article discusses arc reversals in hybrid Bayesian networks with deterministic variables. Hybrid Bayesian networks contain a mix of discrete and continuous chance variables. In a Bayesian network representation, a continuous chance variable is said to be deterministic if its conditional distributions have zero variances. Arc reversals are used in making inferences in hybrid Bayesian networks and influence diagrams. We describe a framework consisting of potentials and some operations on potentials that allows us to describe arc reversals between all possible kinds of pairs of variables. We describe a new type of conditional distribution function, called partially deterministic, if some of the conditional distributions have zero variances and some have positive variances, and show how it can arise from arc reversals.

1. Introduction

Hybrid Bayesian networks are Bayesian networks (BNs) containing a mix of discrete (countable) and continuous (real-valued) chance variables. Shenoy [23] describes a new technique for "exact" inference in hybrid BNs using mixture of Gaussians. This technique consists of transforming a general hybrid BN to a mixture of Gaussians (MoG) BN. Lauritzen and Jensen [12] have described a fast algorithm for making inferences in MoG BN, and it is implemented in Hugin, a commercial software package.

A MoG BN is a hybrid BN such that all continuous variables have conditional linear Gaussian (CLG) distributions, and there are no discrete variables with continuous parents. If we have a general hybrid BN containing a discrete variable with continuous parents, then one method of transforming such a network to a MoG BN is to do arc reversals. If a continuous variable has a non-CLG distribution, then we can approximate it with a CLG distribution. In the process of doing so, we may create a discrete variable with continuous parents. In this case, arc reversals are again necessary to convert the resulting hybrid BN to a MoG BN.

Arc reversals were pioneered by Olmsted [16] for solving discrete influence diagrams. They were further studied by Shachter [19–21] for solving discrete influence diagrams, finding posterior marginals in discrete BNs, and for finding relevant sets of variables for a decision variable in an influence diagram. Kenley [9] generalized arc reversals in influence diagrams with continuous variables having conditional linear Gaussian distributions (see also [22]). Poland [17] further generalized arc reversals in influence diagrams with Gaussian mixture distributions. Recently, Madsen [14] has described solving a class of Gaussian influence diagrams using arc-reversal theory. Although there are currently no exact algorithms to solve general hybrid influence diagrams (containing a mix of discrete and continuous chance variables), a theory of arc reversals is useful in this endeavor. We believe that Olmsted's arc-reversal algorithm for discrete influence diagrams would apply to influence

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diagrams with a mix of discrete, continuous, and deterministic chance variables using the arc-reversal theory described in this paper. This claim, of course, needs further investigation.

Hybrid BNs containing deterministic variables pose a special problem since the joint density for all continuous variables does not exist. Thus, a method for propagating density potentials would need to be modified to account for the non-existence of the joint density [1,3].

A traditional way of handling continuous chance variables in hybrid BNs is to discretize the conditional density functions, and convert continuous nodes to discrete nodes. There are several problems with this method. First, to get a decent approximation, we need to use many bins. This increases the computational effort for computing marginals. Second, even with many bins, based on what evidence is obtained, which may not be easy to forecast, the posterior marginal may result in all mass in one of the bins resulting in an unacceptable discrete approximation of the posterior marginal. One way to mitigate this problem is to do a dynamic discretization as suggested by Kozlov and Koller [11], but this is not as simple as just dividing the sample space of continuous variables evenly into some number of bins.

Another method of handling continuous variables in hybrid BNs is to use mixtures of truncated exponentials (MTEs) to approximate probability density functions [15]. MTEs are easy to integrate in closed form. Since the family of mixtures of truncated exponentials is closed under multiplication, addition, and integration, the Shenoy–Shafer architecture [24] can be used to find posterior marginals. Cobb et al. [4] propose using a non-linear optimization technique for finding mixtures of truncated exponentials approximation for the many commonly used distributions. Cobb and Shenoy [1,2] extend this approach to Bayesian networks with linear and non-linear deterministic variables.

Arc reversals involve divisions. MTEs are not closed under the division operation. Thus, we do not see much relevance for MTEs for describing the arc-reversal theory. However, making inferences in hybrid Bayesian networks with deterministic variables can be described without a division operation, i.e., without arc reversals. And in this case, MTEs can be used to ensure that marginalization of density functions can be easily done.

The main goal of this paper is to describe an arc-reversal theory in hybrid Bayesian networks with deterministic variables. While such a theory would be useful in making inferences in hybrid BNs and also in solving influence diagrams with a mix of discrete, continuous, and deterministic variables, the scope of this paper does not include either inference in hybrid Bayesian networks nor solving influence diagrams.

Arc reversal is described in terms of functions called potentials with combination, marginalization, and division operations. One advantage of this framework is that it can be easily adapted to make inferences in hybrid BNs and to solve hybrid influence diagrams. For example, if we use the Shenoy–Shafer architecture [24] for making inferences in hybrid Bayesian networks, then the potentials that are generated by combination and marginalization operations do not always have probabilistic semantics. For example, the combination of a probability density function and a deterministic equation (which is represented as a Dirac delta function) does not have probabilistic semantics. Nevertheless, as we will show in this paper, the use of potentials is useful for describing arc reversals. Furthermore, we believe that this framework can be extended further for computing marginals in hybrid Bayesian networks and for solving hybrid influence diagrams.

Shachter [20] describes how the arc-reversal theory for discrete Bayesian networks can be used for probabilistic inference. Given that we extend arc-reversal theory for continuous and deterministic variables, Shachter's [20] framework can thus be used for making inferences in hybrid Bayesian networks with deterministic variables. As observed by Madsen [13], an important advantage of using arc-reversal theory for making inferences is that after arc reversal, the network remains a Bayesian network, and we can exploit, e.g., *d*-separation, for probabilistic inference.

An outline of the remainder of this paper is as follows. Section 2 describes the framework of potentials used to describe arc reversals. We use Dirac delta functions to represent conditionally deterministic distributions, and we describe some properties of Dirac delta functions in the Appendix. Section 3 describes arc reversals for arcs between all kinds of pairs of variables. Section 4 describes partially deterministic distributions that arise from arc reversals. Finally, in Section 5, we summarize and conclude.

2. The framework of potentials

In this section we will describe the notation and definitions used in the paper. Also, we will describe a framework consisting of potentials and some operations on potentials that will let us describe the arc-reversal process in hybrid Bayesian networks with deterministic conditionals.

2.1. Variables and states

We are concerned with a finite set V of variables. Each variable $X \in V$ is associated with a set of its possible *states* denoted by Ω_X. If Ω_X is a countable set, finite or infinite, we say X is *discrete*, and depict it by a rectangular node in a graph; otherwise X is said to be *continuous* and is depicted by an oval node.

In a BN, each variable has a conditional distribution function for each state of its parents. A conditional distribution function associated with a continuous variable is said to be *deterministic* if the variances (for each state of its parents) are all zeros. For simplicity, we will refer to continuous variables with non-deterministic conditionals as continuous, and