Nonlinear Analysis: Hybrid Systems 5 (2011) 467–478

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Hybrid extragradient method for general equilibrium problems and fixed point problems in Hilbert space ☆

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ARTICLE INFO

Article history:
Received 22 April 2009
Accepted 20 October 2010

Keywords:
Hybrid extragradient method
Nonexpansive mapping
Variational inequality problem
Equilibrium problem
Fixed points

ABSTRACT

In this paper, we introduce an iterative scheme by the hybrid methods for finding a common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem in a Hilbert space. Then, we prove the strongly convergent theorem by a hybrid extragradient method to the common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of a variational inequality problem. Our results extend and improve the results of Bnouhachem et al. [A. Bnouhachem, M. Aslam Noor, Z. Hao, Some new extragradient iterative methods for variational inequalities, Nonlinear Analysis (2008) doi:10.1016/j.na.2008.02.014] and many others.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle .,.\rangle$ and norm $\|.\|$, respectively, and C is a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ are real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP(F). Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in$ EP(F) if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to find a solution of (2.1) (see [1,2]). In 1997, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when EP(F) is nonempty and proved a strong convergence theorem.

Let $A : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by VI(C, A), is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [3] and the references therein.

Let $B : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem: Find

$$z \in C \quad \text{such that } F(z, y) + \langle Bz, y - z\rangle \geq 0, \ \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by GEP, i.e.,

GEP $= \{z \in C : F(z, y) + \langle Bz, y - z\rangle \geq 0, \forall y \in C\}$. In the case of $B \equiv 0$, GEP is denoted by EP(F).

☆ This research was supported by The Thailand Research Fund and the Commission on Higher Education under grant MRG5380081.
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doi:10.1016/j.nahs.2010.10.005

In the case of $F \equiv 0$, GEP is also denoted by VI(C, A). A mapping A of C into H is called α-inverse-strongly monotone [4] if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$. Recently, Takahashi and Toyoda [5], Yao et al. [6] and Plubtieng and Punpaeng [7] introduced an iterative method for finding an element of VI$(C, A) \cap F(S)$, where $A : C \to H$ is an α-*inverse-strongly monotone* mapping.

We recall that, a mapping $A : C \longrightarrow H$ is said to be *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \text{for all } u, v \in C;$$

A is said to be *k-Lipschitz continuous* if there exists a positive real number k such that

$$\|Au - Av\| \leq k\|u - v\|, \quad \text{for all } u, v \in C;$$

Remark 1.1. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous.

It is well known that if A is a strongly monotone and Lipschitz continuous mapping on C, then the variational inequality problem has a unique solution. How to actually find a solution of the variational inequality problem is one of the most important topics in the study of the variational inequality problem. The variational inequality has been extensively studied in the literature. See, e.g., [3,8] and the references therein.

In 1976, Korpelevich [9] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n) \end{cases} \tag{1.3}$$

for all $n \geq 0$, where $\lambda \in \left(0, \frac{1}{k}\right)$, C is a nonempty closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if VI(C, A) is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.3), converge to the same point $z \in$ VI(C, A).

In 2003, Takahashi and Toyoda [5], introduced the following iterative scheme:

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap$ VI$(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.4) converges weakly to some $z \in F(S) \cap$ VI(C, A). Recently, Zeng and Yao [10] proved the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions: (i) $\lambda_n k \subset (0, 1-\delta)$ for some $\delta \in (0, 1)$ and (ii) $\alpha_n \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$. They proved that the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to the same point $P_{F(S)\cap VI(C,A)}x_0$ provided that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

In 2007, Takahashi et al. [11] introduced the modified Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then we prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In 2008, Bnouhachem et al. [12] introduced the following new extragradient iterative method for finding an element of $F(S) \cap$ VI(C, A). Let C be a closed convex subset of a real Hilbert space H, A be α-inverse strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap$ VI$(C, A) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ be given by

$$\begin{cases} x_1, u \in C \quad \text{chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\alpha_n u + (1 - \alpha_n)P_C(x_n - \lambda_n Ay_n)), \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0, 1)$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common element of $F(S) \cap$ VI(C, A).