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General equilibrium bifunction variational inequalities

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ABSTRACT

In this paper, we introduce and consider a new class of equilibrium variational inequalities, called the mixed general equilibrium bifunction variational inequalities. We suggest and analyze some proximal methods for solving mixed general equilibrium bifunction variational inequalities using the auxiliary principle technique. Convergence of these methods is considered under some mild suitable conditions. Several cases are also discussed. Results in this paper include some new and known results as special cases.

1. Introduction

Variational inequalities, which were introduced and first studied in the sixties, have been seen to be an important and interesting branch of mathematical sciences with applications in industry, regional sciences, and pure and applied sciences. Variational inequalities can be viewed as natural extensions of the variational principles. It is a well-known fact that the optimality conditions for the minimum of a differentiable convex function on a convex set can be characterized by means of the variational inequalities. Noor [1] has shown that the optimality conditions for differentiable nonconvex functions on the nonconvex set can by characterized by means of a class of variational inequalities, called the general variational inequalities. It has been shown that a wide class of odd-order and nonsymmetric boundary value problems can be studied in the unified and general framework of the general variational inequalities. Related to the variational inequalities, there is an equilibrium problem, which is mainly due to Blum and Oettli [2] and Noor and Oettli [3]. Such problems have been studied extensively in recent years due to their importance in pure and applied sciences. These problems have been extended and generalized in several directions using novel and innovative ideas and techniques. See [2,4–10,1,11–18,3,19–32] and the references therein for the applications, formulation, numerical methods and other aspects for the variational inequalities and equilibrium problems.

Motivated and inspired by the research going on in this dynamic and interesting field, we introduce and study a new class of equilibrium problems and bifunction variational inequalities, called the mixed general equilibrium bifunction variational inequalities, in a unified manner. This class includes the mixed general equilibrium problems and the mixed general bifunction variational inequalities as special cases. We have shown that the minimum of a sum of differentiable nonconvex and directionally (Gateaux) differentiable nonconvex functions can be characterized by means of this new class of general equilibrium bifunction variational inequalities. We note that the projection method and its variant form, the resolvent method, cannot be used to suggest some iterative methods for solving the mixed general equilibrium bifunction variational inequalities. This fact motivated us to use the technique of the auxiliary principle of Glowinski et al. [9] to suggest and analyze some implicit iterative methods for solving the mixed general equilibrium bifunction variational inequalities; see Algorithms 3.1–3.3. We also consider the convergence criterion for the proposed method (Algorithm 3.1) under suitable mild conditions, thus obtaining the main results (Theorems 3.1 and 3.2) of this paper. Several special cases of our main results

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are also considered. Results obtained in this paper may be viewed as an improvement and refinement of the previously known results. These may be extended to other classes of variational inequalities and equilibrium problems. Comparison of these methods with other methods is an interesting problem for future research. Readers are encouraged to find novel applications of the general equilibrium bifunction variational inequalities in pure and applied sciences.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let $F(\cdot,\cdot), T(\cdot,\cdot) : H \times H \to R$ be two bifunctions and let $\varphi(\cdot)$ be a continuous function. We recall the following well-known results and concepts.

For given bifunctions $F(\cdot,\cdot), T(\cdot,\cdot) : H \times H \to R$ and nonlinear operator $g : H \to H$, we consider the problem of finding $u \in H$ such that

$$F(g(u), g(v)) + T(u, g(v) - g(u)) + \phi(g(v)) - \phi(g(u)) \geq 0, \quad \forall v \in H. \tag{2.1}$$

An inequality of the type (2.1) is called a mixed general equilibrium bifunction variational inequality. We now show that the minimum of a sum of a differentiable nonconvex function, a directionally (Gateaux) differentiable nonconvex function and a nondifferentiable nonconvex function can be characterized by means of a class of mixed general equilibrium bifunction variational inequalities of type (2.1). For this purpose, we recall the following well-known concepts, which are mainly due to Youness [27] and Noor [18].

Definition 2.1. Let K be subset in the real Hilbert space H. The set K is said to be a g-convex set with respect to the operator $g : H \to H$ if and only if

$$(1-t)g(u) + tg(v) \in K, \quad \forall u, v \in K : g(u), g(v) \in K, t \in [0, 1].$$

It is clear that every convex set is a g-convex set, but the converse is not true; see [18]. Note that if $g \equiv I$, then a g-convex set is a convex set.

Definition 2.2. Let K be a g-convex set in H. A function f on the g-convex set K is said to be a g-convex function if and only if

$$f((1-t)g(u) + tg(v)) \leq (1-t)f(g(u)) + tf(g(v)), \quad \forall u, v \in H : g(u), g(v) \in K, t \in [0, 1].$$

It is known that every convex function is a g-convex function, but the converse is not true; see Noor [18]. For more information, see [11–13,18,27].

Remark 2.1. We now show that the minimum of a sum of differentiable, directionally differentiable and nondifferentiable g-convex functions can be characterized by means of a problem of type (2.1). For this purpose, we consider the functional $I[v]$, which is defined as

$$I[v] = F(v) + f(v) + \varphi(v).$$

If $F(\cdot)$ is a differentiable g-convex function, $f(\cdot)$ is a directionally (Gateaux) differentiable g-convex function and $\varphi(\cdot)$ is a nondifferentiable g-convex function, then, using the technique of Noor [18], one can easily show that the minimum of $I[v]$ on the g-convex set K can be characterized by means of a problem of the type (2.1) with

$$F(g(u), g(v)) = \left\langle F'(g(u), g(v) - g(u))\right\rangle$$

and

$$f'(g(u), g(v) - g(u)) = T(u, g(v) - g(u)).$$

This shows that a problem of the type (2.1) can be used to characterize the optimum of nonconvex (g-convex) functions. Similarly, other problems, which arise in different branches of pure and applied sciences, can be studied via the general and unified framework of the problem (2.1) and its variant forms.

Special cases:

(1) If $F(g(u), g(v)) = 0$, then problem (2.1) is equivalent to finding $u \in H$ such that

$$T(u, g(v) - g(u)) + \phi(g(v)) - \phi(g(u)) \geq 0, \quad \forall v \in H, \tag{2.2}$$

which is known as a mixed general bifunction variational inequality. A wide class of problems arising in elasticity, fluid flow through porous media and optimization can be studied in the general framework of problems (2.2); see [7–9,14,19–23].

If $T(u, g(v) - g(u)) = 0$, then problem (2.1) reduces to the problem of finding $u \in H$ such that

$$F(g(u), g(v)) + \phi(g(v)) - \phi(g(u)) \geq 0, \quad \forall v \in H, \tag{2.3}$$

which is known as a mixed general equilibrium problem considered by Noor [11,13,16] and Noor and Noor [20].