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General equilibrium with heterogeneous participants and discrete consumption times ☆

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ABSTRACT

The paper investigates the term structure of interest rates imposed by equilibrium in a production economy consisting of participants with heterogeneous preferences. Consumption is restricted to an arbitrary number of discrete times. The paper contains an exact solution to market equilibrium and provides an explicit constructive algorithm for determining the state price density process. The convergence of the algorithm is proven. Interest rates and their behavior are given as a function of economic variables.

1. Introduction

Interest rates are determined by the equilibrium of supply and demand. Increased demand for credit brings interest rates higher, while an increase in demand for fixed income investment causes rates to go down. To determine the mechanism by which economic forces and investors' preferences cause changes in supply and demand, it is necessary to develop a general equilibrium model of the economy. Such model provides a means of quantitative analysis of how economic conditions and scenarios affect interest rates.

Vasicek (2005) investigates an economy in continuous time with production subject to uncertain technological changes described by a state variable. Consumption is assumed to be in continuous time, with each investor maximizing the expected utility from lifetime consumption. The participants have constant relative risk aversion, with different degrees of risk aversion and different time preference functions. After identifying the optimal investment and consumption strategies, the paper derives conditions for equilibrium and provides a description of interest rates.

For a meaningful economic analysis, it is essential that a general equilibrium model allows heterogeneous participants. If all participants have identical preferences, then they will all hold the same portfolio. Since there is no borrowing and lending in the aggregate, there is no net holding of debt securities by any participant, and no investor is exposed to interest rate risk. Moreover, if the utility functions are all the same, it does not allow for study of how interest rates depend on differences in investors' preferences.

The main difficulty in developing a general equilibrium model with heterogeneous participants had been the need to carry the individual wealth levels as state

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variables, because the equilibrium depends on the distribution of wealth across the participants. This can be avoided if the aggregate consumption can be expressed as a function of a Markov process, in which case only this Markov process becomes a state variable. This is often simple in models of pure exchange economies, where the aggregate consumption is exogenously specified.

The situation is different in models of production economies. In such economies, the aggregate consumption depends on the social welfare function weights. Because these weights are determined endogenously, it is necessary that the individual consumption levels themselves be functions of a Markov process. This has precluded an analysis of equilibrium in a production economy with any meaningful number of participants; most explicit results for production economies had previously been limited to models with one or two participants.

The above approach is exploited here. Vasicek (2005) shows that the individual wealth levels can be represented as functions of a single process, which is jointly Markov with the technology state variable. This allows construction of equilibrium models with just two state variables, regardless of the number of participants in the economy.

In Vasicek (2005), the equilibrium conditions are used to derive a nonlinear partial differential equation whose solution determines the term structure of interest rates. While the solution to the equation can be approximated by numerical methods, the nonlinearity of the equation could present some difficulties.

The present paper provides the exact solution for the case that consumption takes place at a finite number of discrete times. This solution does not require solving partial differential equations, and explicit computational procedure is provided. If the time points are chosen to be dense enough, the discrete case will approximate the continuous case with the desired precision. Some may in fact argue that, in reality, consumption is discrete rather than continuous, and therefore the discrete case addressed here is the more relevant.

The following section of this paper summarizes the relevant results from Vasicek (2005). Section 3 contains the solution for the equilibrium state price density process and the structure of interest rates in the discrete consumption case. Section 4 gives a proof that the proposed algorithm converges to the market equilibrium.

2. The equilibrium economy

Assume that a continuous time economy contains a production process whose rate of return $\mathrm{d}A/A$ on investment is

$$\frac{\mathrm{d}A}{A}=\mu \mathrm{d}t+\sigma \mathrm{d}y, \tag{1}$$

where $y(t)$ is a Wiener process. The process $A(t)$ represents a constant return-to-scale production opportunity. An investment of an amount W in the production at time t yields the amount $WA(s)/A(t)$ at time $s>t$. The production process can be viewed as an exogenously given asset that is available for investment in any amount. The amount of investment in production, however, is determined endogenously.

The parameters of the production process can themselves be stochastic. It will be assumed that their behavior is driven by a Markov state variable X, $\mu=\mu(X(t),t)$, $\sigma=\sigma(X(t),t)$. The dynamics of the state variable, which can be interpreted as representing the state of the production technology, is given by

$$\mathrm{d}X=\zeta \mathrm{d}t+\psi \mathrm{d}y+\varphi \mathrm{d}x, \tag{2}$$

where $x(t)$ is a Wiener process independent of $y(t)$. The parameters ζ, ψ, and φ are functions of $X(t)$ and t.

It is assumed that investors can issue and buy any derivatives of any of the assets and securities in the economy. The investors can lend and borrow among themselves, either at a floating short rate or by issuing and buying term bonds. The resultant market is complete. It is further assumed that there are no transaction costs and no taxes or other forms of redistribution of social wealth. The investment wealth and asset values are measured in terms of a medium of exchange that cannot be stored unless invested in the production process. For instance, this wealth unit could be a perishable consumption good.

Suppose that the economy has n participants and let $W_k(0)>0$ be the initial wealth of the k–th investor. Each investor maximizes the expected utility from lifetime consumption,

$$\max E\int_0^T p_k(t)U_k(c_k(t))\mathrm{d}t, \tag{3}$$

where $c_k(t)$ is the rate of consumption at time t, $U_k(c)$ is a utility function with $U_k'>0$, $U_k''<0$, and $p_k(t)\geq 0$, $0\leq t\leq T$ is a time preference function. Consider specifically the class of isoelastic utility functions, written in the form

$$\begin{aligned} U_k(c) &= \tfrac{c^{(\gamma_k-1)/\gamma_k}}{\gamma_k-1} \qquad \gamma_k>0,\ \gamma_k\neq 1\\ &=\log c \qquad \gamma_k=1. \end{aligned} \tag{4}$$

Here γ_k is the reciprocal of the relative risk aversion coefficient, $1/\gamma_k=-cU_k''/U_k'$, which will be called the risk tolerance.

An economy cannot be in equilibrium if arbitrage opportunities exist in the sense that the returns on an asset strictly dominate the returns on another asset. A necessary and sufficient condition for absence of arbitrage is that there exist processes $\lambda(t)$, $\eta(t)$, called the market prices of risk for the risk sources $y(t)$, $x(t)$, respectively, such that the price P of any asset in the economy satisfies the equation

$$\frac{\mathrm{d}P}{P}=(r+\beta\lambda+\delta\eta)\mathrm{d}t+\beta \mathrm{d}y+\delta \mathrm{d}x, \tag{5}$$

where β, δ are the exposures of the asset to the two risk sources. In particular,

$$\mu=r+\sigma\lambda. \tag{6}$$

It is assumed that Novikov's condition holds,

$$E\exp\left(\tfrac{1}{2}\int_0^T(\lambda^2+\eta^2)dt\right)<\infty. \tag{7}$$

Let Z be the numeraire portfolio of Long (1990) with the dynamics

$$\frac{\mathrm{d}Z}{Z}=\left(r+\lambda^2+\eta^2\right)\mathrm{d}t+\lambda \mathrm{d}y+\eta \mathrm{d}x, \tag{8}$$