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Rough set theory applied to (fuzzy) ideal theory

John N. Mordeson*

Department of Mathematics and Computer Science, Creighton University, 2500 California Plaza, Omaha, NE 68178, USA

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Abstract

We use covers of the universal set to define approximation operators on the power set of the given set. In Section 1, we determine basic properties of the upper approximation operator and show how it can be used to give algebraic structural properties of certain subsets. We define a particular cover on the set of ideals of a commutative ring with identity in such a way that both the concepts of the (fuzzy) prime spectrum of a ring and rough set theory can simultaneously be brought to bear on the study of (fuzzy) ideals of a ring. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

In 1982, Pawlak introduced the concept of a rough set [18]. This concept is fundamental to the examination of granularity in knowledge. It is a concept which has many applications in data analysis. The idea is to approximate a subset of a universal set by a lower approximation and an upper approximation in the following manner. A partition of the universe is given. The lower approximation is the union of those members of the partition contained in the given subset and the upper approximation is the union of those members of the partition which have a nonempty intersection with the given subset. It is well known that a partition induces an equivalence relation on a set and vice versa. The properties of rough sets can thus be examined via either partitions or equivalence relations. The members of the partition (or equivalence classes) can be formally described by unary set-theoretic operators [27], or by successor functions for upper approximation spaces [7,8]. This axiomatic approach allows not only for a wide range of areas in mathematics to fall under this approach, but also a wide range of areas to be used to describe rough sets. Some examples are topology, (fuzzy) abstract algebra, (fuzzy) directed graphs, (fuzzy) finite state machines, modal logic, interval structures [7,14,15,17,19,27–29]. One may generalize the use of partitions or equivalence relations to that of covers or relations [17,20,22,24,25,29].

In this paper, we use covers of the universal set to define approximation operators on the power set of the given set. In Section 1, we determine basic properties of the upper approximation operator and show how it can be used to give algebraic structural properties

* Tel.: +1-402-280-2418; fax: +1-402-280-5738.
E-mail address: mordes@creighton.edu (J.N. Mordeson).

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of certain subsets. Section 1 lays the ground work for our main results which appear in Section 2. In Section 2, we define a particular cover on the set of ideals of a commutative ring with identity in such a way that both the concepts of the (fuzzy) prime spectrum of a ring [6], and rough set theory can simultaneously be brought to bear on the study of (fuzzy) ideals of a ring. The notion of the (fuzzy) prime spectrum of a ring generalizes that of affine varieties, where the study of polynomial equations occurs. The notion of a fuzzy subset is of course due to Zadeh [30], and a fuzzy substructure of an algebraic structure is due to Rosenfeld [21].

1. Upper and lower approximations defined by covers

Let V be nonempty set and let $\mathscr{P}(V)$ denote the power set of V. Let s be a function of $\mathscr{P}(V)$ into itself. We are interested in the following conditions on s since they are the ones that hold for upper approximation operators defined via an equivalence relation:

(u1) $\forall X \in \mathscr{P}(V),\ X \subseteq s(X)$.

(u2) $\forall X, Y \in \mathscr{P}(V),\ X \subseteq Y \Rightarrow s(X) \subseteq s(Y)$.

(u3) $\forall X, Y \in \mathscr{P}(V),\ s(X \cup Y) = s(X) \cup s(Y)$.

(u4) $\forall X \in \mathscr{P}(V),\ s(X) = s(s(X))$.

Definition 1.1. Let $\mathscr{C} \subseteq \mathscr{P}(V)\backslash\{\emptyset\}$. Then $\mathscr{C}$ is called a *cover* of V if $V \subseteq \bigcup_{C\in\mathscr{C}} C$. Suppose that $\mathscr{C}$ is a cover of V. Then

(i) $\mathscr{C}$ is said to be *semi-reduced* or *semi-irredundant* if $\forall C, D \in \mathscr{C}, C \subseteq D \Rightarrow C = D$;

(ii) $\mathscr{C}$ is said to be *reduced* or *irredundant* if $\forall C \in \mathscr{C}$, there does not exist $\mathscr{C}' \subseteq \mathscr{C}\backslash\{C\}$ such that $C \subseteq \bigcup_{D\in\mathscr{C}'} D$.

Definition 1.2. Let $\mathscr{C}$ be a cover of V. Define $\bar{s} : \mathscr{P}(V) \to \mathscr{P}(V)$ by $\forall X \in \mathscr{P}(V)$, $\bar{s}(X) = \{y \in V \mid \exists C \in \mathscr{C},\ y \in C \text{ and } C \cap X \neq \emptyset\}$. Then $\forall X \in \mathscr{P}(V)$, $\bar{s}(X)$ is called an *upper approximation* of X with respect to $\mathscr{C}$. An upper approximation $\bar{s}$ is said to be *transitive* if $\forall x, y, z \in V$, $x \in \bar{s}(\{y\})$ and $y \in \bar{s}(\{z\})$ imply $x \in \bar{s}(\{z\})$

Proposition 1.3. *Let $\mathscr{C}$ be a cover of V. Then the following properties hold.*

(1) $\forall x \in V,\ x \in \bar{s}(\{x\})$ (*$\bar{s}$ is reflexive*);

(2) $\forall x, y \in V,\ x \in \bar{s}(\{y\}) \Rightarrow y \in \bar{s}(\{x\})$ (*$\bar{s}$ is symmetric*);

(3) $\forall x \in V,\ \forall C \in \mathscr{C},\ C \cap \{x\} \neq \emptyset \Rightarrow C \subseteq \bar{s}(\{x\})$;

(4) (u1)–(u3) *hold.*

Proof. (1) Let $x \in V$. Since $\mathscr{C}$ is a cover of V, $\exists C \in \mathscr{C}$ such that $x \in C$. Hence $C \cap \{x\} \neq \emptyset$. Thus $x \in \bar{s}(\{x\})$ by the definition of $\bar{s}$.

(2) Suppose $x \in \bar{s}(\{y\})$. Then $\exists C \in \mathscr{C}$ such that $x \in C$ and $C \cap \{y\} \neq \emptyset$. Hence $y \in C$ and $C \cap \{x\} \neq \emptyset$. Thus $y \in \bar{s}(\{x\})$.

(3) Suppose $C \cap \{x\} \neq \emptyset$. Then $\forall y \in C,\ y \in \bar{s}(\{x\})$ since $C \cap \{x\} \neq \emptyset$. Hence $C \subseteq \bar{s}(\{x\})$.

(4) Let $x \in X$. Then $\exists C \in \mathscr{C}$ such that $x \in C$ and clearly $C \cap X \neq \emptyset$. Hence $x \in \bar{s}(X)$. Thus (u1) holds. For (u2), let $z \in \bar{s}(X)$. Then $\exists C \in \mathscr{C}$ such that $z \in C$ and $C \cap X \neq \emptyset$. Hence $C \cap Y \neq \emptyset$ and so $z \in \bar{s}(Y)$. Consider (u3). Now $z \in \bar{s}(X \cup Y) \Leftrightarrow \exists C \in \mathscr{C}$ such that $z \in C$ and $C \cap (X \cup Y) \neq \emptyset \Leftrightarrow \exists C \in \mathscr{C}$ such that $z \in C$ and $(C \cap X) \cup (C \cap Y) \neq \emptyset \Leftrightarrow$ either $z \in \bar{s}(X)$ or $z \in \bar{s}(Y) \Leftrightarrow z \in \bar{s}(X) \cup \bar{s}(Y)$. □

We are also interested in the following two conditions on a function $s : \mathscr{P}(V) \to \mathscr{P}(V)$:

(u5) $\forall X \in \mathscr{P}(V)$ and $x \in V$, $x \in s(X)$ implies $\exists$ a finite subset $X' \subseteq X$ such that $x \in s(X')$.

(u6) $\forall \{X_\alpha \mid \alpha \in \Omega\} \subseteq \mathscr{P}(V)$, $s(\bigcup_{\alpha\in\Omega} X_\alpha) = \bigcup_{\alpha\in\Omega} s(X_\alpha)$.

Condition (u5) plays an important role in determining structure results for algebraic structures [32].

When V is finite, condition (u3) plays a major role, but when V is infinite we often need condition (u6). The argument in Proposition 1.3 which shows that condition (u3) holds for $\bar{s}$ is immediately adaptable to show that condition (u6) holds for $\bar{s}$.

Example 1.4. Let $V = \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ denotes the positive integers. Define $s : \mathscr{P}(V) \to \mathscr{P}(V)$ as follows: $\forall X \in \mathscr{P}(V)$, if X is infinite, then $s(X) = X \cup \{\infty\}$ and if X is finite, then $s(X) = X$. Clearly s satisfies conditions (u1) and (u2). We now show that s satisfies (u3). Let $X, Y \in \mathscr{P}(V)$.

Suppose that X and Y are finite. Then $X \cup Y$ is finite and so $s(X \cup Y) = X \cup Y = s(X) \cup s(Y)$.

Suppose that either X is infinite or Y is infinite. Then $X \cup Y$ is infinite. Hence $s(X \cup Y) = X \cup Y \cup \{\infty\} = s(X) \cup s(Y)$. We now show that s satisfies condition (u4). Let $X \in \mathscr{P}(V)$. Suppose that X is finite.