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The negative sign and exponential expressions: Unveiling students' persistent errors and misconceptions

Richard Cangelosi[a,*], Silvia Madrid[a], Sandra Cooper[a], Jo Olson[b], Beverly Hartter[c]

[a] Department of Mathematics, Washington State University, United States
[b] Department of Education, Washington State University, United States
[c] Department of Mathematics, Oklahoma Wesleyan University, United States

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ABSTRACT

The purpose of this study was to determine whether or not certain errors made when simplifying exponential expressions persist as students progress through their mathematical studies. College students enrolled in college algebra, pre-calculus, and first- and second-semester calculus mathematics courses were asked to simplify exponential expressions on an assessment. Persistent errors are identified and characterized. Using quantitative and qualitative methods, we found that the concept of negativity played a prominent role in most of the students' errors. We theorize that an underdeveloped conception of additive and multiplicative inverses is the root of these errors.

1. Introduction

Algebra provides the foundation for advanced mathematical thinking; and proficiency in algebraic manipulations is essential to students who want to enter science, technology, engineering, and mathematics (STEM) careers (Liston & O'Donoghue, 2010). Research on the development of algebraic reasoning is an emerging focus area in mathematics education (e.g., Kieran, 2007; Seng, 2010; Vlassis, 2002a, 2002b; Warren, 2003). Most studies focus their attention on functions (e.g., Dugdale, 1993; Thompson, 1994; Vinner, 1992) or solving linear equations (e.g., Sfard & Linchevski, 1994; Slavit, 1997). Comparatively few studies investigate the simplification of algebraic expressions (Ayres, 2000; Sakpakornkan & Harries, 2003), a skill which requires students to use their understanding of variables and to interpret mathematical symbols accurately. In addition, research on students' understanding of the negative sign is limited, particularly in the context of exponential notation (Kieran, 2007).

This study grew out of a week-long workshop with approximately 40 high school juniors and seniors. During the workshop, which focused on exponential and logarithmic expressions and equations, it became obvious that students had a fragile understanding of exponential expressions. The researchers currently teach university level courses, and the errors made by the high school students seemed remarkably similar to those committed by university level students. This led to the conjecture that when working with exponential expressions there are persistent errors, namely errors that students continue to make as they progress through more advanced courses. In this article, by students we mean a collection of individuals, not an individual student whose progress is tracked.

* Corresponding author. Tel.: +1 509 335 3926; fax: +1 509 335 1188.
E-mail address: rcangelosi@wsu.edu (R. Cangelosi).

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This lead us to the current study were we investigate the following two research questions:

(1) Are there persistent errors made when students simplify exponential expressions as students progress with their mathematical studies?
(2) If persistent errors exist, what are they and how can they be categorized?

Through this study we discovered that there are persistent errors and these are frequently associated with a negative sign in the expression. We conclude the paper by offering conjectures as to the source of these errors.

2. Theoretical framework

This study used an investigative approach based on a constructivist perspective in which the process of learning requires the learner to adapt existing knowledge from previous experiences to accommodate new ideas. The focus of research from the constructivist framework places primacy on the individual and how knowledge is constructed. To understand the development of students' knowledge regarding exponential expressions, we use a framework proposed by Sfard (1991, 1992) which builds upon the notions of concept image and concept definition (Tall & Vinner, 1981).

A mathematical concept is a complex web of ideas developed from mathematical definitions and mental constructs (Sfard, 1991, 1992; Tall & Vinner, 1981; Vinner, 1992). Sfard uses the term concept to mean a mathematical idea within "...the formal universe of ideal knowledge," and the term conception to represent "...the whole cluster of internal representations and associations evoked by the concept." (1991, p. 3). We adopt Sfard's framework because it provides a detailed model of the process of learning mathematics.

Sfard's (1991, 1992) theoretical model for the learning of mathematical concepts encompasses both operational (procedural, algorithmic) understanding and structural (conceptual, abstract) understanding, characterizing both as necessary and complementary. According to Sfard (1991), when learning a new concept, a natural starting point is through a definition. Some mathematical definitions treat concepts as objects that exist and are components of a larger system. This is considered a structural conceptualization. On the other hand, concepts can also be defined in terms of processes, algorithms, or actions leading to an operational conception. A structural conception requires the ability to visualize the mathematical concept as a "real thing" that exists as part of an abstract mathematical structure, whereas an operational conception implies more of a potential that requires some action or procedure to be realized.

Sfard emphasizes that the operational and structural conceptions are not mutually exclusive; they are complementary. The two aspects of conception can be considered as two sides of the same coin; both are critical to building a deep understanding of mathematics. Based on historical examples and cognitive theory, she asserts that when learning new mathematics, the natural entry point is through an operational approach. She claims that the precedence of the operational aspects of conception over the structural aspects is an invariant characteristic of learning "...which appear[s] to be quite immune to changes in external stimuli" (1991, p. 17). The transition from an operational understanding to a structural understanding occurs through stages and is a long and "inherently difficult" process.

As students move from an operational to a structural understanding, they go through three stages: *interiorization*, *condensation*, and *reification*. During the first stage, interiorization, the student becomes skilled at performing processes involving the concept until these processes can be carried out mentally and with ease. For example, an individual may start with the concept of inverse operations as "the opposite of addition is subtraction" and "the opposite of multiplication is division." At this stage students can use this concept to solve basic linear equations. However, they may not recognize the role of the additive and multiplicative identities in the process.

During the second stage, condensation, the learner is able to think about a complicated process as a whole without needing to carry out the details. The person is able to break the process into manageable units without losing sight of the whole. In this stage, there is also a growing facility with moving between different representations, recognizing similarities, and making connections. This stage lasts as long as the mathematical notion remains tied to certain processes. For example, students may recognize zero as the additive identity, one as the multiplicative identity, and the role of the identity elements. By recognizing the similarities between additive and multiplicative inverses, students begin to see both as specific examples of the concept of inverse.

A concept is reified when the student can perceive the concept as an object and use it as an input to develop more advanced ideas (Sfard, 1991, 1992). Reification represents a significant shift in thinking, one in which the concept is suddenly seen as part of a larger mathematical structure. It is at this stage that students begin to operate with a concept as an object and as the input into new processes. In fact, reification frequently requires being exposed to more advanced concepts which require this new object as a building block. For example, the concept of inverse applied to numbers may be reified when students need to extend the concept of inverse to functions. The stage of reification is the most difficult and often happens as a flash of insight (Sfard, 1991).

We hypothesize that if persistent errors exist when simplifying exponential expressions, the associated concept development may be stalled at an operational stage. Characterization of the persistent errors will help us identify confounding elements and begin to understand how to move students to a structural stage.