



Purchasing life insurance to reach a bequest goal



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ARTICLE INFO

Article history:

Received February 2014

Received in revised form

May 2014

Accepted 19 July 2014

Available online 1 August 2014

Keywords:

Term life insurance

Whole life insurance

Bequest motive

Deterministic control

Non-forfeiture value

ABSTRACT

We determine how an individual can use life insurance to meet a bequest goal. We assume that the individual's consumption is met by an income from a job, pension, life annuity, or Social Security. Then, we consider the wealth that the individual wants to devote towards heirs (separate from any wealth related to the afore-mentioned income) and find the optimal strategy for buying life insurance to maximize the probability of reaching a given bequest goal. We consider life insurance purchased by a single premium, with and without cash value available. We also consider irreversible and reversible life insurance purchased by a continuously paid premium; one can view the latter as (instantaneous) term life insurance.

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1. Introduction

Life insurance helps in estate planning, specifically, in providing bequests for children, grandchildren, or charitable organizations. With this purpose in mind, we determine how an individual can use life insurance to meet a bequest goal. We assume that the individual's consumption is met by an income from a job, pension, life annuity, or Social Security. Then, we consider the wealth that the individual wants to devote towards heirs (separate from any wealth related to the afore-mentioned income) and find the optimal strategy for buying life insurance to maximize the probability of reaching a given bequest goal.

In this paper, we join two hitherto unconnected streams of literature. The first stream is that of optimal purchasing of life insurance, and most of the research in this area maximizes utility of consumption, bequest, or both. The seminal article in this area is Richard (1975); please see Bayraktar and Young (2013) for some recent references relevant to the problem of maximizing utility of household consumption by using life insurance.

The second stream is that of maximizing the probability of reaching a particular target. This problem has been studied in probability problems related to gambling, as in Dubins and Savage (1965, 1976). For an important extension of the work of Dubins and Savage, see Pestien and Sudderth (1985), in which they control a diffusion process to reach a target before ruining. For related

papers, see Sudderth and Weerasinghe (1989), Kulldorff (1993), and Browne (1997, 1999a,b). Instead of controlling a diffusion, we maximize the probability of reaching a particular goal and allow the individual to purchase life insurance to help reach that goal, while adding a random deadline (namely, death).

The rest of the paper is organized as follows: In Section 2.1, we consider the case for which the individual buys whole life insurance via a single premium with no cash value available, while in Section 2.2, she can surrender any or all of her whole life insurance for a cash value. In both cases, we compute her expected wealth at death because her goal is to reach a given bequest, so expected wealth at death is relevant. Section 3 parallels Section 2 for the case in which insurance is purchased via a continuously-paid premium; however, we reverse the order of the topics as compared with the order in Section 2. In Section 3.1, the individual is allowed to change the amount of her insurance at any time; in our time-homogeneous setting, this amounts to instantaneous term life insurance. By contrast, in Section 3.2, we do not allow the individual to terminate life insurance, so for the remainder of her life, she has to pay for any life insurance she buys. The solution of the problem in Section 3.1 is simpler than and informs the solution to the problem in Section 3.2, so we present the simpler problem first. Section 4 concludes the paper.

2. Single-premium life insurance

We begin this section by stating the optimization problem that the individual faces. In Section 2.1, we consider the case for which the individual buys whole life insurance via a single premium with

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no cash value available, so she never surrenders her life insurance policy; she may only buy more. In Section 2.2, we incorporate a non-zero cash value and find the optimal insurance purchasing and surrendering policies in that case. At the end of each of Sections 2.1 and 2.2, we compute her expected wealth at death.

2.1. No cash value available

We assume that the individual has an investment account that she uses to reach a given bequest goal b . This account is separate from the money that she uses to cover her living expenses. The individual may invest in a riskless asset earning interest at the continuous rate $r > 0$, which actuaries call the *force of interest*, or she may purchase whole life insurance.

Denote the future lifetime random variable of the individual by τ_d . We assume that τ_d follows an exponential distribution with mean $\frac{1}{\lambda}$. (In other words, the individual is subject to a constant *force of mortality*, or hazard rate, λ .) The individual buys life insurance that pays at time τ_d . This insurance acts as a means for achieving the bequest motive. In this time-homogeneous model, we assume that a dollar death benefit payable at time τ_d costs H at any time. Write the single premium as follows:

$$H = (1 + \theta)\bar{A}_x = (1 + \theta)\frac{\lambda}{r + \lambda},$$

in which $\theta \geq 0$ is the proportional risk loading. Assume that θ is small enough so that $H < 1$; otherwise, if $H \geq 1$, then the buyer would not pay a dollar or more for one dollar of death benefit.

In this section and in Section 2.2, we suppose that the premium is payable at the moment of the contract; as stated above, H is the single premium per dollar of death benefit. In Section 3, we consider the case for which the insurance premium is payable continuously.

Let $W(t)$ denote the wealth in this separate investment account at time $t \geq 0$. Let $D(t)$ denote the amount of death benefit payable at time τ_d purchased at or before time $t \geq 0$. Thus, with single-premium life insurance, wealth follows the dynamics

$$\begin{cases} dW(t) = rW(t-)dt - H dD(t), & 0 \leq t < \tau_d, \\ W(\tau_d) = W(\tau_d-) + D(\tau_d-). \end{cases}$$

An insurance purchasing strategy $\mathbf{D} = \{D(t)\}_{t \geq 0}$ is *admissible* if (i) \mathbf{D} is a non-negative, non-decreasing process, independent of τ_d , and (ii) if wealth under this process is non-negative for all $t \geq 0$. We include the latter condition to prevent the individual from borrowing against her life insurance.

Remark 2.1. By requiring that \mathbf{D} be non-decreasing over time, we effectively assume the individual cannot surrender any life insurance once she has bought it. In the real world, whole life insurance has a surrender value that the individual can withdraw, and in Section 2.2, we include that feature. □

We assume that the individual seeks to maximize the probability that $W(\tau_d) \geq b$, by optimizing over admissible controls \mathbf{D} . The corresponding value function is given by

$$\phi(w, D) = \sup_{\mathbf{D}} \mathbf{P}^{w, D}(W(\tau_d) \geq b), \tag{2.1}$$

in which $\mathbf{P}^{w, D}$ denotes conditional probability given $W(0-) = w \geq 0$ and $D(0-) = D \geq 0$. We call ϕ the maximum probability of reaching the bequest goal.

If $D \geq b$, then the individual has already reached her bequest goal of b ; thus, henceforth, in this section, we assume that $D < b$. If wealth equals $H(b - D)$, the so-called *safe level*, then it is optimal for the individual to spend all of her wealth to purchase life insurance of $b - D$ so that her total death benefit becomes $b = (b - D) + D$. It

follows that $\phi(w, D) = 1$ for $w \geq H(b - D)$ and $0 \leq D < b$. Thus, it remains only to determine the maximum probability of reaching the bequest on $\mathcal{R} = \{(w, D) : 0 \leq w \leq H(b - D), 0 \leq D < b\}$.

We next present a verification lemma that states that a “nice” solution to a variational inequality associated with the maximization problem in (2.1) is the value function ϕ . Therefore, we can reduce our problem to one of solving a variational inequality. We state the verification lemma without proof because its proof is similar to others in the literature; see, for example, Wang and Young (2012a,b) for related proofs in a financial market that includes a risky asset.

Lemma 2.1. Let $\Phi = \Phi(w, D)$ be a function that is non-decreasing and differentiable with respect to both w and D on $\mathcal{R} = \{(w, D) : 0 \leq w \leq H(b - D), 0 \leq D < b\}$, except that Φ might have infinite derivative with respect to w at $w = 0$. Suppose Φ satisfies the following variational inequality on \mathcal{R} , except possibly when $w = 0$:

$$\max(rw \Phi_w - \lambda \Phi_w, \Phi_D - H \Phi_w) = 0. \tag{2.2}$$

Additionally, suppose $\Phi(H(b - D), D) = 1$. Then, on \mathcal{R} ,

$$\phi = \Phi. \quad \square$$

The region $\mathcal{R}_1 = \{(w, D) \in \mathcal{R} : \phi_D(w, D) - H \phi_w(w, D) < 0\}$ is called the *continuation* region because when the wealth and life insurance benefit lie in the interior of \mathcal{R}_1 , the individual does not purchase additional life insurance; rather, she *continues* with her current benefit and invests her wealth in the riskless asset. Indeed, $\phi_D < H \phi_w$ means that the marginal benefit of buying more life insurance (ϕ_D) is less than the marginal cost of doing so ($H \phi_w$). On the closure of that region in \mathcal{R} , written $cl(\mathcal{R}_1)$, the equation $\lambda \phi = r w \phi_w$ holds.

To help us solve the variational inequality (2.2), we recall that in similar problems (for example, purchasing life annuities to minimize the probability of lifetime ruin, as described in Milevsky et al., 2006), the optimal strategy is to “act” only at the safe level. In our case, that translates into buying life insurance only when wealth reaches $H(b - D)$, so that ϕ solves the following boundary-value problem for $0 \leq w \leq H(b - D)$ and $0 \leq D < b$:

$$\begin{cases} r w \phi_w - \lambda \phi = 0, \\ \phi(H(b - D), D) = 1. \end{cases} \tag{2.3}$$

Buying life insurance only when wealth reaches $H(b - D)$ is indeed optimal, as we prove in the following proposition.

Proposition 2.2. The maximum probability of reaching the bequest goal on $\mathcal{R} = \{(w, D) : 0 \leq w \leq H(b - D), 0 \leq D < b\}$ is given by

$$\phi(w, D) = \left(\frac{w}{H(b - D)} \right)^{\frac{\lambda}{r}}. \tag{2.4}$$

The associated optimal life insurance purchasing strategy is not to purchase additional life insurance until wealth reaches the safe level $H(b - D)$, at which time, it is optimal to buy additional life insurance of $b - D$.

Proof. We use Lemma 2.1 to prove this proposition. First, note that ϕ in (2.4) is increasing and differentiable with respect to both w and D on \mathcal{R} , except possibly at $w = 0$. Because ϕ solves the boundary-value problem (2.3), we have $r w \phi_w - \lambda \phi = 0$ on \mathcal{R} .

Next, we show that $\phi_D - H \phi_w \leq 0$ on \mathcal{R} :

$$\begin{aligned} & \phi_D(w, D) - H \phi_w(w, D) \\ &= \frac{\lambda}{rH} \left(\frac{w}{H(b - D)} \right)^{\frac{\lambda}{r}-1} \left[\frac{w}{(b - D)^2} - \frac{H}{b - D} \right] \\ & \propto w - H(b - D) \leq 0. \end{aligned}$$

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